MATHEMATICS

ON THE NUMBER OF POSITIVE INTEGERS $\leqslant x$ AND FREE OF PRIME FACTORS $> y$

BY

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(Dedicated to Prof. L. E. J. Brouwer on his 70th birthday)

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Notations and conventions. For $x > 0$, $y \geqslant 2$, $\Psi(x, y)$ denotes the number of integers specified in the title.

The ratio $(\log x)/\log y$ is always denoted by u.

γ denotes EULER's constant.

For definition of $\varrho(u)$ see (1, 2); for $\Lambda(x, y)$ see (2. 9); for $R(y)$ see (3. 7) and (3. 8). $\pi(y)$ denotes the number of primes $\leqslant y$; li y denotes the logarithmic integral

$$\lim_{\delta \to 0} \{ \int_0^{1-\delta} dt/\log t + \int_{1+\delta}^{y} dt/\log t \}.$$

C_i denotes a positive constant; $C_j(z)$ denotes a positive function of z.

§ 1. *Introduction.* Recently we studied the asymptotic behaviour of $\Phi(y, x)$, denoting the number of positive integers $\leqslant x$ and free of prime divisors $< y$ (see [2]). In the present paper we shall apply the same methods to $\Psi(x, y)$, the definition of Ψ being obtained from that of Φ by replacing "$< y$" by "$> y$". The problems of Φ and Ψ are analogous in many respects, although the latter seems to be the more difficult one [1]).

The problem of $\Psi(x, y)$ was discussed before by S. D. CHOWLA and T. VIJAYARAGHAVAN [4], V. RAMASWAMI [5] and A. A. BUCHSTAB [1] [2]).

The smallest $O-$ term was given by RAMASWAMI. Our improvements on RAMASWAMI's results are the following:

1. We have an error term of the type $O\{x(u+1)^2 R(y)\}$ or $O\{x(\log y)^2 R(y)\}$ (see (1. 3) and (5. 3)), where RAMASWAMI had, for k arbitrary but fixed, $O\{x(\log y)^{-k}\}$. This improvement has been made possible by the construction of $\Lambda(x, y)$ (see § 2).

[1]) The reason is the following one. Both $\Phi(x, y)$ and $\Psi(x, y)$ have a trivial first approximation, and the errors have the same nature in both cases. The difficulty lies, of course, in the discussion of the error terms. The first approximation to Φ is large, and hence we are apt to be content with a rough estimate of the error. For Ψ, however, the first approximation is zero, and therefore a detailed discussion of the "error" becomes essential, and of course we do not call it the "error".

[2]) BUCHSTAB considered the number of integers in a given arithmetic progression, which are free of primes $> y$. This problem can be tackled by our methods as well. (See [2], p. 806).

2. For very large values of u (for instance $u \geqslant \log y$), $x\,R(y)$ becomes much larger than Ψ itself. In that case, formulas (1, 6) and (1. 7) still give acceptable estimates.

If u is fixed, $x = y^u$ then the behaviour of $\Psi(x, y)$ is comparatively simple. We have (see [1, 4, 5] [3]))

$$(1.\ 1) \qquad \lim_{y\to\infty} y^{-u}\,\Psi(y^u, y) = \varrho(u),$$

where $\varrho(u)$ is defined by the following properties

$$(1.\ 2) \quad \begin{cases} \varrho(u) = 0 \quad (u < 0) & ; \quad \varrho(u) = 1 \quad (0 \leqslant u \leqslant 1)\ ; \\ u\,\varrho'(u) = -\varrho(u-1) \quad (u > 1) & ; \quad \varrho(u) \text{ continuous for } u > 0. \end{cases}$$

The function $\varrho(u)$ was closely investigated in [3]; for the main results of that paper we refer to (2. 7), (4. 3) and (1. 8) below.

By (1. 1), $x\varrho(u)$ is, in some sense, an approximation to $\Psi(x, y)$. We shall replace it by a closer approximation, viz. $\Lambda(x, y)$, defined by (2. 9). In § 2, this formula will be obtained heuristically, and afterwards, (2. 9) will serve as a definition.

Our main result, to be proved in § 3, is

$$(1.\ 3) \qquad |\,\Psi(x,y) - \Lambda(x,y)\,| < C_1\, x u^2\, R(y) \qquad (x > 0,\ y \geqslant 2).$$

$R(y)$ is, roughly, the order of $|\pi(y) - li\ y|/y$ (see (3. 7) and (3. 8)).

It can be seen from (1. 6) and (1. 7), that (1. 3) is definitely worthless if u is large. This will certainly be the case if $u \geqslant \log y$ [4]). For $0 < x \leqslant y$, that is for $u \leqslant 1$, the left-hand-side of (1. 3) is equal to zero.

In § 4, the behaviour of $\Lambda(x, y)$ will be studied. The main result is [5]) [6])

$$(1.\ 4) \quad |\,\Lambda(x,y) - x\,\varrho(u)\,| < C_2 + \frac{C_3\, x\,\varrho(u)\log(2+u)}{\log y} \quad (x > 1,\ y \geqslant 2,\ 0 < u < \log y).$$

For u fixed, $u \gtrless 0$ we infer

$$(1.\ 5) \qquad \Lambda(y^u, y)/y^u \to \varrho(u) \qquad (y \to \infty).$$

Formula (1. 3) will be proved by a method which is based upon a recurrence relation for $\Psi(x, y)$ (see (3. 1)).

An entirely different method by which $\Psi(x, y)$ can be studied lies in interpreting the problem of $\Psi(x, y)$ as the coefficient problem of a certain Dirichlet series, and using the well-known methods for that problem.

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[3]) The result (1. 1) will not be used in the present paper; it follows independently from (1. 3) and (1. 5) below.

[4]) This is of course not due to the factor u^2 in (1. 3). Actually, this factor has comparatively small influence, and therefore we do not bother about removing it. It can be replaced by $(\log y)^2$ (see (5. 3)).

[5]) It might be possible to consider a larger domain than $0 < u < \log y$. However, we need not go into this trouble, since for $u > \log y$ the error term in (1. 3) is much larger than $\Psi(x, y)$ itself.

[6]) If $1 \leqslant u \leqslant \log y$, the term C_2 can be omitted (see (1. 8)).

This was carried out for $\Phi(x, y)$ in [2], § 3. The same can be done for $\Psi(x, y)$. The only alteration is a simplification; we have to replace

$$\zeta(s) \prod_{p<y} (1-p^{-s}) \quad \text{by} \quad \prod_{p\leqslant y} (1-p^{-s})^{-1}.$$

Not bothering about the unimportant improvements which are made possible by this simplification, we obtain

(1. 6) $\quad \Psi(x,y) < C_4\, x \log^2 y\; e^{-u\log u - u \log\log u + C_5 u} \qquad (3<u<4\, y^{1/2}/\log y;\ y \geqslant 2),$

(1. 7) $\quad \Psi(x,y) < C_6\, x^{2/3} \qquad (u > 4\, y^{1/2}/\log y;\ y \geqslant 2).$

For the details we refer to [2].

It may be remarked that the exponential in (1. 6) does not fall very far from $\varrho(u)$. Actually we have (see [3], § 1)

(1. 8) $\quad \varrho(u) = \exp\{-u \log u - u \log\log u + O(u)\} \qquad (u > 3).$

It is not difficult to obtain, from (1. 3), (1. 6) and (1. 7), a simple rough inequality giving at least something for all values of x and y simultaneously:

(1. 9) $\quad \Psi(x,y) < C_7\, x\, e^{-C_8 u} \qquad (x > 1,\ y \geqslant 2).$

It may be remarked that (1. 7) becomes relatively weak if u is much larger than $4\, y^{1/2}/\log y$. Actually we have (see § 2)

(1. 10) $\quad \Psi(x,y) < C_9(y)\, (\log x)^{\pi(y)} \qquad (x \geqslant y \geqslant 2).$

For y fixed, the order is much less than $x^{2/3}$.

In § 5, the contents of the present paper will be applied to a special problem.

§ 2. *Construction of* $\Lambda\ (x, y)$

If we put

$$\prod_{p\leqslant y} (1-p^{-s})^{-1} = \sum_{n=1}^{\infty} a_n(y)\, n^{-s} \qquad (\operatorname{Re} s > 0),$$

then we have (cf. [3], p. 805)

$$\Psi(x,y) = \sum_{n \leqslant x} a_n(y),$$

and hence, by a formula of RIEMANN, if x is not an integer,

(2. 1) $$\Psi(x,y) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^s \prod_{p\leqslant y} (1-p^{-s})^{-1} \frac{ds}{s} \qquad (a > 0).$$

Concerning this formula we make a remark which is not connected with the further contents of this section. If y is fixed, and x tends to infinity, then we have the asymptotic formula (cf. [5], p. 101)

(2. 2) $$\Psi(x,y) \sim A_r\, (\log x)^r,$$

where $r = \pi(y)$, and $A_r = \{r!\, \prod_{p\leqslant y} \log p\}^{-1}$. This is easily seen by interpretation of $\Psi(x, y)$ as the number of lattice points in a certain r-dimensional simplex. On the other hand, it can be expected that the residue of the integral (2. 1) at the major pole $s = 0$ will furnish a good approximation to $\Psi(x, y)$. This residue is a polynomial in $\log r$ of degree r, whose first term is indeed equal to $A_r(\log x)^r$, and probably this polynomial is a better approximation than (2. 2).

If, however, we consider large values of y, the behaviour of $\Psi(x, y)$ is totally different from (2. 2) [7]). This can be expected more or less from the fact that, for large y, the strip $0 < \mathrm{Re}\, s < \frac{1}{2}$ in the integral (2. 1) is practically inpenetrable. Furthermore it can be expected that the behaviour of the integrand of (2. 1) near the point $s = 1$ will become important.

Putting

$$(2.\ 3)\qquad \log \prod_{p>y} (1 - p^{-s}) = - \int_y^\infty \frac{dt}{t^s \log t} + P(y, s) \qquad (\mathrm{Re}\, s > 1),$$

we have $P(y, s) \to 0$ as $y \to \infty$, uniformly for any bounded sub-set of the half-plane $\mathrm{Re}\, s > 1$. This follows from the well-known fact $\pi(y) - \mathrm{li}\, y = O\{y(\log y)^{-2}\}$. Furthermore we have, for $\mathrm{Re}\, s > 1$,

$$- \int_y^\infty \frac{dt}{t^s \log y} = \int_{-\infty}^{(1-s)\log y} \frac{e^t}{t}\, dt.$$

In order to obtain an approximation to (2. 1) we shall neglect the error term $P(y, s)$ in (2. 3), and so we replace (2. 1) formally by

$$(2.\ 4)\qquad \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} x^s \zeta(s) \exp\left\{ \int_{-\infty}^{(1-s)\log y} \frac{e^t}{t} dt \right\} \frac{ds}{s}.$$

Substituting $s = 1 - w/\log y$, and using the identity

$$(2.\ 5)\qquad w \exp\left\{ \gamma + \int_0^w \frac{e^t - 1}{t} dt \right\} = - \exp\left\{ \int_{-\infty}^w \frac{e^t}{t} dt \right\},$$

the integral (2. 4) is easily transformed into

$$(2.\ 6)\qquad \frac{x}{2\pi i} \int_{-i\infty}^{i\infty} e^{-uw} K\left(- \frac{w}{\log y}\right) \exp\left\{ \gamma + \int_0^w \frac{e^t - 1}{t} dt \right\} dw,$$

where $K(z)$ stands for $z\zeta(1 + z)/(1 + z)$, and $u = (\log x)/\log y$.

In [3] it was proved that

$$(2.\ 7)\qquad \varrho(u) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \exp\left\{ -uw + \gamma + \int_0^w \frac{e^t - 1}{t} dt \right\} dw \qquad (u \neq 0).$$

In order to express (2. 6) formally in terms of the function ϱ, we shall express the factor $K(-w/\log y)$ as a Laplace integral $\int e^{\lambda w} f(\lambda)\, d\lambda$. We have, indeed,

$$(2.\ 8)\qquad K(z) = \frac{z\,\zeta(1+z)}{1+z} = \int_0^\infty t^{-z}\, d\frac{[t]}{t} \qquad (\mathrm{Re}\, z > -1),$$

[7]) Unless, of course, x exceeds some very rapidly increasing function of y.

and so, using (2. 7) we formally transform (2. 6) into

$$\Lambda(x, y) = x \int_0^\infty \varrho\left(\frac{\log x - \log t}{\log y}\right) d\, \frac{[t]}{t}. \tag{2. 9}$$

We shall take this as the definition of $\Lambda(x, y)$. Since $\varrho(\lambda) = 0$ for $\lambda < 0$ and $\varrho(\lambda)$ is continuous everywhere apart from the jump at $\lambda = 0$, the integral in (2. 8) is well-defined, unless the jump of ϱ coincides with a jump of $[t]/t$. This will be the case if x is a positive integer n; in that case (2. 8) is ambiguous. This does not matter very much. In the first place we have $\Lambda(n + 0, y) - \Lambda(n - 0, y) = 1$ $(n = 1, 2, 3, \ldots)$, which is relatively small. Secondly, in the sequal we are able to exlude integral values of x without essentially demaging our results. For the sake of completeness, however, we shall define

$$\Lambda(n, y) = \Lambda(n + 0, y). \tag{2. 10}$$

It can be shown that the integrals (2. 4) and (2. 6) above are convergent for non-integral values of x, and that the transformation into (2. 8) is legitimate. We do not go into that, since in our presentation (2. 4) and (2. 6) serve heuristic purposes only.

It may be remarked that, in (2. 8), the lower limit 0 of the integral may be replaced by any number a $(0 \leqslant a < 1)$, and that the upper limit ∞ may be replaced by any number b $(b \geqslant x)$.

§ 3. *Proof of* (1. 3)

We shall show in this section that $\Psi(x, y) - \Lambda(x, y)$ is relatively small. This method is the same as method B in [2]. Throughout this section we assume $x > 0$, $y \geqslant 2$.

It follows from the definition of $\Psi(x, y)$, that we have the functional equation

$$\Psi(x, y) = \Psi(x, y^h) - \sum_{y < p \leqslant y^h} \Psi\left(\frac{x}{p}, p\right) \qquad (h \geqslant 1). \tag{3. 1}$$

Furthermore we have

$$\Psi(x, y) = [x] \qquad (0 \leqslant x \leqslant y). \tag{3. 2}$$

The function $\Psi(x, y)$ is completely determined by (3. 1) and (3. 2). For, applying (3. 1), we can evaluate it successively in the regions $y < x \leqslant y^2$, $y^2 < x \leqslant y^3$, etc., by taking $h = u = (\log x)/\log y$.

The function $\Lambda(x, y)$ satisfies an equation which is similar to (3. 1). It arises from (3. 1) by writing the sum in (3. 1) formally as a Stieltjes integral $\int \Psi(x/\mu, \mu)\, d\pi(\mu)$ and replacing $\pi(\mu)$ by $li\ \mu$. We have, indeed,

$$\Lambda(x, y) = \Lambda(x, y^h) - \int_y^{y^h} \Lambda\left(\frac{x}{\mu}, \mu\right) \frac{d\mu}{\log \mu} \qquad (h \geqslant 1). \tag{3. 3}$$

This follows from (2. 9) by the identity

(3. 4) $$x\varrho\left(\frac{\log x-\log t}{\log y}\right)=x\varrho\left(\frac{\log x-\log t}{\log y^h}\right)-\int_y^{y^h}\frac{x}{\mu}\varrho\left(\frac{\log\frac{x}{\mu}-\log t}{\log\mu}\right)\frac{d\mu}{\log\mu},$$

which holds for all $h \geqslant 1$ and $t > 0$. This one is easily proved by partial differentiation with respect to h, in virtue of (1. 2).

The identity (3. 4) fails to prove (3. 3) if x is an integer; in that case we have to use (2.10).

A second analogy between Ψ and Λ is given by

(3. 5) $$\Psi(x, y) = \Lambda(x, y) \qquad (0 < x \leqslant y).$$

For, it immediately follows from (2. 9) and (2. 10) that, for $0 < x \leqslant y$, we have $\Lambda(x, y) = x\int_0^x d([t]/t) = [x]$.

We shall now estimate the difference

(3. 6) $$\Delta(x, y; h) = \sum_{y<p\leqslant y^h}\Psi\left(\frac{x}{p}, p\right) - \int_y^{y^h}\Psi\left(\frac{x}{\mu}, \mu\right)\frac{d\mu}{\log\mu}.$$

To this end we introduce a function $R(y)$ satisfying

(3. 7) $$R(y) \downarrow 0 \quad (y\to\infty) \quad , \qquad R(y) > y^{-1}\log y \quad (y \geqslant 2),$$

(3. 8) $$|\pi(y) - li\, y| < y\, R(y)/\log y \qquad (y \geqslant 2).$$

We can take, for instance $R(y) = C_{10}\exp\{-C_{11}(\log y)^{1/2}\}$.

$\Psi(x, y)$ represents, by its definition, the number of a set of positive integers n. Let n be a positive integer, and let q be its largest prime factor (if $n = 1$, we take $q = 1$). Then n gives the contribution 1 to $\Psi(x, y)$ if and only if $n \leqslant x$, $q \leqslant y$. Hence its contribution to $\Psi(x\mu^{-1}, \mu)$ is 1 if and only if μ satisfies $q \leqslant \mu \leqslant x/n$. Let $\Delta_n(x, y; h)$ denote the contribution of n to $\Delta(x, y; h)$, then we have

(3. 9) $$\Delta(x, y; h) = \sum_{1\leqslant n\leqslant x/y}\Delta_n(x, y; h).$$

Let J denote the intersection of the intervals $q \leqslant \mu \leqslant x/n$ and $y < \mu \leqslant y^h$. Clearly, $\Delta_n = 0$ if J is empty. If J is not empty, then we have

$$\Delta_n(x, y; h) = \sum_{p\in J}1 - \int_J\frac{d\mu}{\log\mu},$$

that is, Δ_n equals $[\pi(\mu) - li\,\mu]_a^b + \beta$ if a and b denote the end-points of J. The unimportant term β is 1 if J is of the type $[p, \beta]$, where p is a prime, and $\beta = 0$ otherwise. We now easily infer from (3. 7) and (3. 8), since both a and b are $\geqslant y$,

$$|\Delta_n| \leqslant 4\, y^h R(y)/\log y \qquad (n \leqslant xy^{-h});$$

$$|\Delta_n| \leqslant 4\, xn^{-1} R(y)/\log y \qquad (xy^{-h} < n \leqslant xy^{-1}).$$

Now (3. 9) gives

$$(3.10) \qquad |\Delta(x,y;h) \leqslant C_{12}\, x\, R(y)\, \{h-1+(\log y)^{-1}\}.$$

We put, for $y \geqslant 2$, $k=1, 2, \ldots,$

$$(3.11) \qquad s_k(y) = \sup_{0<x\leqslant t^k,\ t\geqslant y} x^{-1}\,|\Psi(x,t)-\Lambda(x,t)|,$$

and, by induction with respect to k, we shall prove inequalities for $S_k(y)$ which at the same time show that $s_k(y) < \infty$ for all k and y. Putting $\Psi - \Lambda = \Xi$, we infer from (3. 1), (3. 3) and (3.6) that

$$(3.12) \quad \Xi(x,y) = \Xi(x,y^h) - \int_y^{y^h} \Xi\left(\frac{x}{\mu},\mu\right)\frac{d\mu}{\log\mu} - \Delta(x,y;h) \qquad (h \geqslant 1).$$

We take an integer $k \geqslant 2$, and we apply (3. 12) with $h = k/(k-1)$, $x \leqslant y^k$. Then we have $x \leqslant (y^h)^{k-1}$, and for $y \leqslant \mu \leqslant y^h$ we have $x/\mu \leqslant \mu^{k-1}$. Therefore it follows from (3. 11) that

$$|\Xi(x,y)| \leqslant s_{k-1}(y)\left\{x + \int_y^{y^h} \frac{x}{\mu}\frac{d\mu}{\log\mu}\right\} + C_{12}\, x\, R(y)\left\{\frac{1}{k-1}+\frac{1}{\log y}\right\},$$

and so

$$(3.13)\ s_k(y) \leqslant s_{k-1}(y)\left\{1+\log\frac{k}{k-1}\right\} + C_{12}\, R(y)\left\{\frac{1}{k-1}+\frac{1}{\log y}\right\}\ (k=2,3,\ldots).$$

For $k=1$ we have, by (3. 5), $s_1(y)=0$. Finally, using $\log(k/(k-1)) > (k-1)^{-1}$, we can deduce from (3. 13), by induction, that

$$k^{-1}\, s_k(y) \leqslant (k-1)^{-1}\, s_{k-1}(y) + C_{12}\, R(y)\,\{k^2-k)^{-1} + (k\log y)^{-1}\},$$

whence

$$s_k(y) \leqslant k\, C_{12}\, R(y)\,\{C_{13} + (1+\tfrac{1}{2}+\ldots+(1/k))/\log y\} \qquad (k=1,2,\ldots)$$

The right-hand-side is $O\{k^2 R(y)\}$. By (3. 11) (and 3. 5), this leads to (1. 3).

§ 4. *Asymptotic behaviour of $\Lambda(x,y)$*

Our first aim is to prove (1. 4)

We shall need the following auxiliary formulas,

$$(4.1) \qquad 0 > \varrho'(u)/\varrho(u) > -C_{14}\log(u+1) \qquad (u>1),$$

$$(4.2) \qquad 0 > \varrho''(u)/\varrho'(u) > -C_{15}\log u \qquad (u>2).$$

These are easily established from the asymptotic formula for $\varrho(u)$ viz.

$$(4.3) \quad \varrho(u) \backsim (2\pi u)^{-1/2}\exp\left\{\gamma - u\xi + \int_0^\xi \frac{e^s-1}{s}\,ds\right\} \qquad (u\to\infty),$$

where ξ is the positive solution of $e^\xi - 1 = u\xi$ (see [3]), and using the equation $u\varrho'(u) = -\varrho(u-1)$ in order to express ϱ' and ϱ'' in terms of ϱ. We can even obtain $\varrho'(u) \sim -\varrho(u)\log u$, $\varrho''(u) \sim -\varrho'(u)\log u$. Now (4. 1) and

(4. 2) follow from the fact that the functions $\varrho(u)$, $-\varrho'(u+1)$, $\varrho''(u+2)$ are positive and continuous for $u \geqslant 0$ (see (1. 2)).

We shall tackle $A(x, y)$ by partial integration. We have, by (2. 9),

$$A(x,y) = x \int_{1-}^{x+} \varrho\left(\frac{\log x - \log t}{\log y}\right) d\, \frac{[t]-t}{t} = O(1) + x \int_{1-}^{x-} =$$

$$= O(1) + x\left[\varrho\left(\frac{\log x - \log t}{\log y}\right) \frac{t-[t]}{t}\right]_{1-}^{x-} + x \int_{1}^{x} \varrho'\left(\frac{\log x - \log t}{\log y}\right) \frac{[t]-t}{t^2 \log y}\, dt,$$

and therefore

$$(4.\ 4) \quad |A(x,y) - x\varrho(u)| < C_{16} + x \int_{1}^{x} \left|\varrho'\left(\frac{\log x - \log t}{\log y}\right)\right| \frac{dt}{t^2 \log y} \quad (y > 2, u > 0).$$

From (4. 2) we infer

$$\varrho'(u-s)/\varrho'(u) < C_7\, u^{C_{15} s} \qquad (u > 2,\ 0 < s \leqslant u),$$

bearing in mind that $\varrho'(\lambda)/\varrho(2)$ is bounded for $0 < \lambda \leqslant 2$. Therefore, if $u > 2$ the integral in (4. 4) is less than

$$|\varrho'(u)| \int_{1}^{x} C_{17}\, t^{C_{15}(\log u)/(\log y)} (t^2 \log y)^{-1}\, dt.$$

If $C_{15} \log u < \frac{1}{2} \log y$, the exponent of t is less than $\frac{1}{2}$, and we infer, using (4. 1)

$$(4.\ 5) \qquad |A(x, y) - x\varrho(u)| < C_{16} + C_{18}\, x\varrho(u) \log(u+2) (\log y)^{-1}.$$

The same thing is easily proved if the restriction $u > 2$ is replaced by $u > 0$. For, if $0 < v \leqslant u \leqslant 2$, the ratio $\varrho'(v)/\varrho(u)$ is uniformly bounded, and so (4. 4) again leads to (4. 5).

On the other hand, if $C_{15} \log u > \frac{1}{2} \log y$, but $u < \log y$, $y \geqslant 2$, then only a bounded region of the u-y-plane is under consideration. In that region, the left-hand-side of (4. 5) is bounded. Therefore, (4. 5) holds, with a constant C_{19} instead of C_{16}, for $0 < u < \log y$, $y \geqslant 2$. This proves (1. 4).

The process of partial integration carried out above can of course be repeated. We then arrive at an expansion which was announced (for Ψ instead of A) by Ramaswami ([5], p. 109). We can give the terms of the expansion explicitly, and we can improve Ramaswami's O-term. Our method results in the following formula:

$$(4.\ 6) \quad \begin{cases} \left| A(x, y) - x \sum_{\nu=0}^{n} \frac{a_\nu \varrho^{(\nu)}(u)}{(\log y)^\nu} \right| < C_{20}(n) + C_{21}(n) \frac{x\varrho(u) (\log(2+u))^{n+1}}{(\log y)^{n+1}} \\ \qquad\qquad\qquad\qquad\qquad\qquad (n+1 < u < \log y,\ y \geqslant 2). \end{cases}$$

Here n is an arbitrary integer $\geqslant 0$, and the coefficients a_ν are taken from the power series $K(z) = a_0 + a_1 z + a_2 z^2 + \ldots$ (see (2. 8)).

These coefficients could be expected from the expression (2. 6), which can be written formally as

$$x\,K\left(\frac{1}{\log y}\,\frac{d}{du}\right)\varrho(u).$$

We do not go into a detailed dissussion of (4. 6). For, if u is large, the only useful information on $\varrho(u)$ we have is the *asymptotic* formula (4. 3), and asymptotically the sum accurring at the left of (4. 6) is equivalent to $x\varrho(u)$ itself.

§ 5. *Application to the average of the logarithm of the largest prime divisor of n.* Let $g(n)$ denote the largest prime dividing n, $(n = 2, 3, \ldots)$, and put $g(1) = 1$. Chowla and Vijayaraghavan [4] showed $g(x) \geqslant h(x)$ for almost all numbers n in the interval $l \leqslant n \leqslant x$, if $h(x)$ is any function satisfying $\log h(x) = o(\log x)$. From the results of the present paper more definite information on $g(x)$ can be obtained. We shall prove that

$$\sum_{n\leqslant x} \log g(n) = a\,x \log x + O(x), \tag{5. 1}$$

where the constant a is given by

$$a = \int_0^\infty (1+u)^{-2}\,\varrho(u)\,du. \tag{5. 2}$$

We shall first prove the following modification of (1. 3):

$$|\Psi(x, y) - \Lambda(x, y)| < C_{22}\,x\,(\log y)^2\,R(y) \qquad (x > 1,\ y \geqslant 2). \tag{5. 3}$$

We may assume $x \geqslant y$, for otherwise we have $\Psi = \Lambda$, by (3. 5).
It is easily deduced from (4. 4), using $\varrho'(\lambda) = 0\,(\exp - \frac{1}{4}\lambda)$, that

$$|\Lambda(x, y) - x\varrho(u)| < C_{16} + C_{23}\,x\,e^{-\frac14 u}/\log y \qquad (x > 1,\ y \geqslant 2), \tag{5. 4}$$

and so, by (1. 8)

$$|\Lambda(x, y)| < C_{16} + C_{24}\,x\,e^{-\frac14 u} \qquad (x > 1,\ y \geqslant 2)$$

Hence it follows, by (1. 9),

$$|\Psi - \Lambda| < C_7\,x\,e^{-C_8 u} + C_{16} + C_{24}\,x\,e^{-\frac14 u} \leqslant C_{16} + C_{25}\,x\,e^{-C_{26} u} \qquad (x > 1, y \geqslant 2).$$

If we assume furthermore $C_{26}\ u \geqslant \log y$, and $x \geqslant y$, then we infer, using (3. 7),

$$|\Psi - \Lambda| < C_{16} + C_{27}\,x\,y^{-1} < C_{28}\,x\,(\log y)^2\,R(y).$$

Therefore, (5. 3) has been proved in this case. If, on the other hand, $C_{26}\,u \leqslant \log y$, then (5. 3) directly follows from (1. 3).

Using a Stieltjes integral, we may write

$$\sum_{n\leqslant x} \log g(n) = \int_{y=2}^{y=x} \log y\; d\,\Psi(x, y) + O(\log x) \qquad (x > 2); \tag{5. 5}$$

the O-term arises from the terms with $n = 2, 2^2, 2^3, \ldots$.

In (5. 5) we shall replace Ψ by Λ. To this end we first consider

$$\int^{x} \log y \, d\{\Psi(x,y) - \Lambda(x,y)\} =$$

$$= [(\log y)\{\Psi(x,y) - \Lambda(x,y)\}]_2^x - \int_2^x y^{-1}\{\Psi(x,y) - \Lambda(x,y)\}\, dy.$$

Here we have $\Psi(x, 2) = O(\log x)$, $\Lambda(x, 2) = O(x)$ (see (5. 4)), and the other contributions are easily seen to be $O(x)$, if we use (5. 3) with $R(y) = \exp\{-(\log y)^{1/2}\}$. Consequently

$$\sum_{n \leqslant x} \log g(n) = \int_{y=2}^{y=x} \log y \, d\, \Lambda(x,y) + O(x). \tag{5. 6}$$

For simplicity we shall assume that x is not an integer, in order to exclude the ambiguity in the definition of Λ (see § 2). In the final results we can easily get rid of this restriction.

It follows from (3. 3), that $\Lambda(x, y)$ is a continuous function of y, and that [8])

$$\frac{\partial \Lambda(x,y)}{\partial y} = \Lambda\left(\frac{x}{y}, y\right) \Big/ \log y.$$

Consequently, the integral in (5. 6) equals

$$\int_2^x \Lambda\left(\frac{x}{y}, y\right) dy. \tag{5. 7}$$

We have, by (5. 4), that

$$\int_2^x \Lambda\left(\frac{x}{y}, y\right) dy = O(x) + \int_2^x \frac{x}{x}\left\{\varrho\left(\frac{\log(x/y)}{\log y}\right) + O\left(\frac{1}{\log y}\exp\left(-\frac{\log(x/y)}{4\log y}\right)\right)\right\} dy.$$

Substitution of $y = x^{1/s}$ in the integrals leads to

$$x \int_1^{(\log x)/\log 2} \{\varrho(s-1)\, s^{-2} \log x + O(s^{-1} e^{-\frac{1}{4}s})\}\, ds.$$

Since $\varrho(s) = O(e^{-s})$ it follows that (5. 7) is

$$\int_2^x \Lambda\left(\frac{x}{y}, y\right) dy = a x \log x + O(x),$$

where a is defined by (5. 2). Now (5. 1) follows from (5. 6).

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[8]) The derivative does not exist at points where x/y is an integer. At these points the derivative has a finite jump.

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