REVIEW OF THE MOVIE ON THE MATHEMATICAL GENIUS RAMANUJAN

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The Man Who Knew Infinity

Directed by Matt Brown

Produced by Edward R. Pressman Film, Xeitgeist Entertainment Group, Animus Films, American Entertainment Investors, Kreo Films FZ

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Jeremy Irons as G. H. Hardy, Dev Patel as Ramanujan

The Man Who Knew Infinity is a film (referred as INFINITY from here on) about the remarkable life story and pathbreaking work of Srinivasa Ramanujan, a self taught genius from India, who communicated bewildering mathematical formulae in two letters in 1913 to the esteemed G. H. Hardy of Cambridge University, England, a towering figure in the mathematical domains of analysis and number theory. The legend is that the Hindu Goddess Namagiri came in Ramanujan's dreams and gave him these formulae. Realizing that Ramanujan was a genius on par with Euler and Jacobi, Hardy invited Ramanujan to Cambridge where Ramanujan did pioneering work by himself and with Hardy. The asymptotic series for the partition function they jointly obtained [12] is a crowning achievement of their collaboration. The film is based on a book by Robert Kanigel [13] under the same title - a book which gives a complete and accurate account of Ramanujan's life. Starting briefly with Ramanujan's life and initial discoveries in India, the film deals mostly with his life in England - his interactions with Hardy who wants "proofs" for the incredible mathematical claims of Ramanujan, the resistance by some British mathematicians to recognize the significance of Ramanujan's work, Ramanujan's insight in obtaining the asymptotic series for the partition function, the difficulties he experienced in England during the tumultous period of World War I, and the effort made by Hardy to get Ramanujan elected Fellow of the Royal Society (FRS) and Fellow of Trinity College (in that order), which finally bore fruit after the first unsuccessful attempt of election to a Fellowship of Trinity. Ramanujan fell ill in England and returned to India in 1919. He died shortly thereafter in April 1920 at the young age of 32. But the vast number of deep results spanning the fields of analysis, number theory and modular forms that he proved in his brief life, place him among the greatest mathematicians in history. The film concludes with Hardy receiving the news about Ramanujan's demise in India and his emotional address to the Royal Society when he describes Ramanujan's work as profoundly original.

The film does introduce dramatization for effect, thereby deviating from the true story in certain places, but the overall depiction of Ramanujan's life will move the audience to tears. Ramanujan's life story is so fascinating and intriguing, that dramatization is really not needed to get audiences excited. There is a good deal of mathematical discussion in the movie but in a way that the audience will be inspired and not be put off by it.

First I will briefly describe Ramanujan's life and work so that my comments on the movie can be better understood.

Srinivasa Ramanujan (1887-1920)

Ramanujan was born on December 22, 1887, in his maternal grandfather's hometown Erode, in the state of Tamil Nadu, in South India. But he grew up in the nearby town of Kumbakonam where his parents lived. Ramanujan and his family were orthodox Iyengars - a subsect of the Brahmin caste of the Hindu religion. Ramanujan's father was an accountant to a cloth merchant and so his family was poor. His mother Komalatammal was a strong willed woman. Ramanujan's family had a special veneration for Goddess Namagiri of the Temple of Namakkal near Kumbakonam. Komalattammal was childless for many years, and so the family prayed to Namagiri to bless her with children, and shortly thereafter Ramanujan was born as her first child.

Ramanujan showed his unusual mathematical talents early. As a boy, he would wake up in the middle of the night and write down mathematical formulae on a piece of slate that he kept next to his bed, and would record these later in Notebooks that he maintained (photostat copies of these Notebooks were later published [21]). The legend is that the Goddess Namagiri would come in his dreams and give him these formulae which were startling even to professional mathematicians. Ramanujan did not provide any proofs of his formulae in these Notebooks.

Although Ramanujan did well in school, owing to his excessive preoccupation with mathematics, he did poorly in other subjects. So he dropped out of college and therefore did not possess a Bachelors degree. Unable to find individuals in India who could evaluate his work, Ramanujan wrote two letters to G. H. Hardy of Cambridge University giving several samples of spectacular formulae he had discovered, but without any hint of proofs or reasoning to justify such formulae. Hardy in consultation with his distinguished colleague J. E. Littlewood came to the conclusion that Ramanujan was a true genius ranking among the greatest mathematicians in history, and for sheer manipulative ability was rivalled only by Euler and Jacobi. These formulae spanned several important domains in the realms of number theory and analysis. Several formulae in the letters were profoundly original, but there were some that were well known, and others that were incorrect as stated. Hardy felt that Ramanujan's time should not be wasted in rediscovery of past work owing to lack of formal training. So he invited Ramanujan to Cambridge so that his untutored genius could be given a proper sense of direction.

The orthodox Brahmins believed that it was a sin to cross the oceans, and so even though Ramanujan was willing to go to England, his mother would not give him permission. The Goddess of Namakkal seems to have played a role in resolving this problem! One story is that Ramanujan's mother had a dream in which she saw Ramanujan seated in an assembly of "white men" and being honored, and the Goddess of Namakkal ordering her not to stand in the way of her son's recognition!! Was this a premonition of Ramanujan's election to Fellowship of the Royal Society? In any case, when Komalattammal got up in the morning, she gave him permission to sail to England.

Ramanujan was in England only for five years (1914-19), but in this brief period (i) he got a number of theorems he had discovered in India published in prestigious journals, (ii) made several new and fundamental discoveries by himself in England, and (iii) collaborated with Hardy on two major projects, one of which being the asymptotic series for the partition function by the powerful *circle method* that they introduced for the first time.

But conditions in wartime England were tough, and especially difficult of Ramanujan who as a strict vegetarian could not easily find food to suit his needs, and who did not know how to protect himself from the cold in the winters. Consequently he often fell ill and was in and out of hospitals frequently. In spite of all this, his mathematical creativity and profundity were unbelievably high. So original were his contributions, that Hardy felt he deserved to be elected Fellow of Trinity College and Fellow of the Royal Society (FRS). But Hardy feared that owing to Ramanujan's rapid decline in health, he will not live long. Thus there was a real urgency in recognizing Ramanujan for his outstanding contributions. So he made a stupendous effort to successfully convince his distinguished colleagues to confer these honors on Ramanujan, overcoming the initial resistance to the Fellowship of Trinity proposal. Ramanujan returned to India in 1919 a very sick man. Just prior to that, Hardy wrote to Francis Dewsbury, Registrar of the University of Madras, saying that " he will return with a scientific standing and reputation such as no Indian has enjoyed before" ([10], p. 200). Ramanujan died in Madras in April 1920, but even in his last few months, he made pathbreaking discoveries, and wrote one last letter to Hardy outlining his discovery of the mock theta functions of orders 3, 5, and 7. These are now considered to be among Ramanujan's deepest contributions.

In the final months of his life in Madras, India, Ramanujan used to ask his wife Janaki Ammal for loose sheets of paper to write down his new found results. After Ramanujan's death, his wife (who did not even have a school education) had the good sense to collect these loose sheets and deliver them to the University of Madras from where they were sent to Hardy. Eventually these sheets came to the possession of G. N. Watson in Birmingham, the world's premier authority in the field of special functions. Watson analysed the results on mock theta functions of order 3 in Ramanujan's last letter to Hardy, and presented his findings as his Retiring Presidential Address to the London Mathematical Society. But there was much more on these loose sheets than what was summarized by Ramanujan in his letter. After Watson's death, these loose sheets were put in storage in the Watson Estate in the Wren Library in Cambridge University. But the world forgot about them, and so these sheets acquired the name "The Lost Notebook" of Ramanujan. It was George Andrews who unearthed The Lost Notebook in 1976 and wrote several papers on its contents in the next decade (see [5] for instance); ever since, the mathematics in the Lost Notebook has been the focus of research in many of the leading centres around the world. The published form of the Lost Notebook [22] was released during the Ramanujan Centennial in 1987.

Ramanujan's work is not only characterized by profound originality, but also that new unexpected connections are revealed between apparently disparate fields. Ramanujan typically wrote down the most striking or significant case of a general result. An identity of Ramanujan is like the tip of an iceberg. Investigation of his identities reveal vast theories that underlie them, like the mass of the iceberg underneath water. Ramanujan's work has had deep impact on several fields within mathematics like number theory, analysis, combinatorics, and the theory of modular forms, to name a few, and to domains outside of mathematics like computer science and physics. To be more specific, Ramanujan's mathematics has strongly influenced the following fields:

(i) Hypergeometric and q-hypergeometric series

(ii) Partitions and combinatory analysis

(iii) Additive number theory via the circle method
(iv) Probabilistic number theory
(v) Elliptic and theta functions
(vi) Modular forms and automorphic functions
(vii) Special functions and definite integrals
(viii) Continued fractions
(ix) Diophantine equations
(x) Irrationality and transcendence
(xi) Fourier analysis
(xii) Lie algebras
(xiii) Statistical mechanics and conformal field theory in physics
(xiv) Computer science and computer algebra

This list is not complete and is definitely growing. The Ramanujan Journal launched in 1997 by Kluwer and now published by Springer, is devoted to all areas of mathematics influenced by Ramanujan including the list of topics given above (see [2], pp. 147-151, for an article on the conception and need of this journal).

Hardy felt that the real tragedy of Ramanujan was not his early death, but that in his formative years he wasted much time on rediscovery. Hardy argued that in mathematics one's best work is done at a young age. He held the view that had Ramanujan lived longer, he would have discovered more theorems, but not results of greater originality. But Hardy may have been wrong in this estimate because Ramanujan's work on mock theta functions in the few months before his death revealed that he was still on the rise in terms of originality. But Hardy did give Ramanujan the highest score for pure talent as reported by Paul Erdös, one of the most influential mathematicians of the 20-th century: Rating mathematicians on a scale of 1 to 100 for pure talent, Hardy gave himself a score of 25, his colleague Littlewood a score of 30, the great German mathematician Hilbert a score of 80, and Ramanujan a perfect score of 100!

Even though we understand Ramanujan's work so much more now than at Hardy's time, we have not the faintest idea how his mind worked. Thus he is an intriguing character to study, so fascinating, that he has inspired several stage productions, the latest being the film under review.

The book, "The Man Who Knew Infinity"

Robert Kanigel's book *The Man Who Knew Infinity* was published by Charles Scribners in 1991. It is a detailed and accurate portrayal of Ramanujan's life. The book also is about Hardy, and therefore may be viewed as a double biography. Kanigel describes beautifully and with emotion all the intriguing aspects of Ramanujan's life (see my review [1]). There have been biographies of Ramanujan published earlier - by Ragami [17] in the Tamil language, and by S. R. Ranganathan [18] in English, to mention just two. Kanigel's book is far more comprehensive compared to earlier biographies, but he does rely on both Ranganathan and Ragami for certain facts and events. My only criticism of Kanigel's book is that he spends too much time discussing Hardy's homosexual inclinations without any concrete evidence. Hardy was a confirmed bachelor - as were many Cambridge dons totally consumed by their work - and the comment Hardy made, namely "my association with Ramanujan was the one romantic incident in my life" is really to be taken in the spirit of mathematical romance, and not in sexual terms.

Kanigel gathered information about Ramanujan during his visit to India in 1987 for the Ramanujan Centennial when mathematicians around the world gathered there to pay homage to the Indian genius. It was an appropriate time to assess the impact Ramanujan's work has had on various parts of mathematics and to discuss the expanding sphere of Ramanujan's mathematical influence in the future. Kanigel used the Ramanujan Centennial not only to talk to the international group of mathematical experts, but also to discuss with many Indian mathematicians and persons who were familiar with Ramanujan's life and family. By the time Ramanujan's 125-th birth anniversary was celebrated in 2012, Kanigel's book had, not surprisingly, an enormous impact the world over, and was being translated into several languages including Tamil. Even though this movie based on Kanigel's book was conceived more than a decade ago, the production picked up steam around the time of the Ramanujan 125-th birthday celebrations just as the Ramanujan Centennial was crucial for Kanigel's effort on his biography of Ramanujan.

The movie, "The Man Who Knew Infinity"

It is almost impossible to produce a movie about mathematics that provides an accurate account, is dignified, and would still appeal to the public at large. People in general have a strong antipathy towards mathematics, and view mathematical research as needless abstraction. But Ramanujan's life story has all the drama, excitement, and intrigue, to attract audiences. Sadly, the general public is turned on by foul language, sex, or violence, and movie producers often introduce these to whet their appetite and ensure the success of the production. For example, the movie *Goodwill Hunting*, which is the story of a mathematical genius without a sense of purpose in life, is full of foul language and it portrays geniuses as freaks rather than those who can expand the horizon of our knowledge. But Goodwill Hunting was a box office hit even though I felt that the producers failed to utilize an unusually interesting story to convey the importance and excitement of mathematics done by brilliant minds. In contrast, even though INFINITY is not a documentary, it depicts various exciting aspects of Ramanujan's life and contributions tastefully; the producers and the Director have to be congratulated for this. Having eminent mathematicians Ken Ono (Emory University) and Manjul Bhargava (Princeton University) as Associate Producers, has ensured that whatever mathematics is discussed or presented is accurate and significant. There are of course several instances in INFINITY where dramatizations for the sake of effect have been introduced, or variations in the theme have been made, and I will discuss some of them in the sequel. These can be accepted as "artists licence" and in no way do these detract from the quality of the production. But some of these variations and dramatizations are either unnecessary or could mislead the audience into thinking that certain events happened when they never really did.

To play the role of G. H. Hardy, the experienced British actor Jeremy Irons was chosen and he does a superb job. Hardy was only in his thirties when Ramanujan met him in 1914; although Jeremy Irons has a striking resemblance to Hardy, he looks much older in the movie than Hardy was at that time.

The popular Indian actor Dev Patel of *Slumdog Millionaire* fame, plays the role of Ramanujan. Patel is much taller, slimmer, and more fit than Ramanujan ever was, but he is a good actor who plays the role Ramanujan quite well. A good looking internationally known actor will definitely accomplish the goal of selling the Ramanujan character to the public around the world, and so the choice of Dev Patel is understandable even though he bears little resemblance to Ramanujan.

Even though the movie is based on Kanigel's book which is a complete biography, the movie focuses on Ramanujan's life in England. The film does begin with a brief portrayal of Ramanujan in India prior to his departure to England. One of the early scenes is Ramanujan and his wife *Janaki* entering their new home in Madras to set up life together. Ramanujan was a college dropout and had difficulty getting a job to support himself. So his parents thought that if they could get him married, he would become more responsible and would seriously begin working like normal people. Thus Ramanujan was married to Janaki in 1909, when she was barely nine years old! Yes, she was a child when she got married - child marriages were common in India at that time - but she stayed in her parents home until she reached puberty, and joined Ramanujan only then. In 1912, Ramanujan was working as a clerk at the Madras Port Trust, and so he and Janaki began life together in Madras. The Indian-American actress Devika Bhise plays the role of Janaki. Wisely, by introducing Janaki in the movie as a mature girl, the producers have avoided any depiction of the child marriage which would have distracted the worldwide public to a discussion of this primitive practice in Hindu society instead of focussing on the brilliant achievements of Ramanujan.

In traditional Hindu society, the girl after marriage moves into the boy's home, and this often means living with the in-laws. So in the scene when Ramanujan arrives in a bullock cart at his new residence in Madras with Janaki, he is accompanied by his mother. Komalattammal's dominant character is so aptly portrayed by the senior actress Arundhathi Nag, that you actually would end up hating her when you see the movie! The movie shows Janaki writing letters to Ramanujan while he was in England and as a dutiful daughter-in-law giving these letters to Komalattammal to mail them to Ramanujan; but Komalattammal hid these letters instead of mailing them, and likewise she hid the letters that Ramanujan wrote to Janaki instead of handing them to her. The great aggravation and anxiety this caused to both Ramanujan and Janaki is emphasized in the movie. What we know is that a letter Ramanujan wrote to Janaki informing her about his return to India in ill health and asking her to meet him on arrival in Bombay, was not delivered to her. Janaki was living in her brother's home at that time and Ramanujan, not knowing this, sent his letter to his mother's address, and she did not forward that to Janaki. Kanigel ([13, p. 318) says that when Ramanujan and Janaki were together in Madras after his return from England, he found out from her that his mother had intercepted their letters.

One of the persons in India who realized that Ramanujan was special as a man and a mathematician was Narayana Iyer with whom Ramanujan discussed his work regularly, and who supervised Ramanujan in his job at the Madras Port Trust. A scene in the movie shows Sir Francis Spring, Narayana Iyer's supervisor, initially rejecting Ramanujan's mathematics as worthless, but with Narayana Iyer's influence, Francis Spring realizes that Ramanujan is a genius. It is true that Narayana Iyer played a key role in convincing Francis Spring how special Ramanujan was, but there are no records to indicate that Ramanujan was initially spurned by Spring. This is one of several instances in the movie where it is shown that Ramanujan was treated poorly by the British, but some of these depictions are fictional and exaggerated.

Even though the ideas were all his, Ramanujan had Narayana Iyer communicate a few mathematical formulas of his to the Indian Mathematical Society instead of submitting them himself. The scenes depicting the discussions between Narayana Iyer and Ramanujan into the late hours of the night convey effectively the total pre-occupation of Ramanujan with his mathematics, and Narayana Iyer's almost parental interest in Ramanujan. But then, Ramanujan fails to spend time with his young wife Janaki. The scene where Janaki comes late at night to where Ramanujan and Narayana Iyer are in discussion, and reminds Ramanujan that he should not forget her, is quite touching.

Mathematics is as much art as it is a science. Pure mathematicians conduct research because they are drawn to the aesthetic beauty of the subject. Indeed Hardy never cared for use or applications of any of his results. This is not to say that mathematics is not useful or applicable, but that "beauty" is really what attracts pure mathematicians to their area of activity. Ramanujan was definitely attracted by the beauty of the subject, and in the movie there is a scene where he tells Janaki that his mathematical equations are as lovely as the flowers in a garden, and that he seeks mathematical patterns like one would marvel at the patterns in nature. Even P. A. M. Dirac, the Nobel Laureate physicist, has stressed that physical laws should have mathematical beauty. The idea that mathematics is beautiful is alien to the lay public, and so the Director of INFINITY has to be complimented for emphasizing the aesthetic aspect as the driving force behind the pursuit of mathematics.

Ramanujan was a loving and caring husband. There is a romantic scene in the movie of Ramanujan and Janaki on the beach in Madras. Ramanujan hailed from an orthodox Hindu family and so would never have been frolicking amorously with his wife in public places. In traditional Hindu society, intimacy is to be shown only in the privacy of your bedroom, and not in public. Thus the scene of Ramanujan and Janaki romancing on the beach is pure artist's fantasy - a scene that the general public will enjoy very much!

Ramanujan shared his family's devotion to the Goddess of Namakkal. According to his Indian biographers Seshu Aiyar (also could be spelled Iyer) and Ramachandra Rao ([24], p. xii), Ramanujan said that the Goddess of Namakkal inspired him with formulae in his dreams. There are several scenes in the movie relating to Ramanujan's religious observances and the inspiration he derives from the Goddess of Namakkal. There is one scene where Ramanujan is writing formule on the stone slabs of a temple floor. The movie also has scenes showing Ramanujan's religious observances in England, and Ramanujan making a statement to Hardy that "An equation to me has no meaning unless it expresses a thought of God". In his biography on Ramanujan, S. R. Ranganathan [18] says Ramanujan made this statement to a friend, but this friend was not Hardy.

The two letters that Ramanujan wrote to Hardy in 1913 communicating dozens of incredible formulae, must go down in history as perhaps the greatest mathematical letters ever written! He begins his letter to Hardy saying "Sir, I beg to introduce myself as a humble clerk in the Accounts Department of the Port Trust Office in Madras ...I have been employing the spare time at my disposal to work at mathematics...I am striking out a new path for myself... The local mathematicians are unable to understand me in my higher flights...if you are convinced that there is anything of value, I would like to have my theorems published... Yours truly S. Ramanujan"

The letters contained formulae for the number of primes up to a given magnitude with some finer statements as to their distribution, some incredible definite integral evaluations, equally stunning continued fractions evaluations, modular identities for elliptic and theta functions, and so on. I provide two examples:

$$\frac{1}{1+}\frac{e^{-2\pi\sqrt{5}}}{1+}\frac{e^{-4\pi\sqrt{5}}}{1+}\cdots = \{\frac{\sqrt{5}}{1+\{5^{3/4}((\sqrt{5}-1)/2)^{5/2}-1\}^{1/5}} - \frac{\sqrt{5}+1}{2}\}e^{2\pi/\sqrt{5}} \tag{1}$$

and

$$\text{the coefficient of } x^n \text{ in } \quad \frac{1}{1-2x+2x^4-2x^9+2x^{16}\cdots} \tag{2}$$

$$\text{is the nearest integer to } \quad \frac{1}{4\pi}\{\cosh \pi\sqrt{n} - \frac{\sinh \pi\sqrt{n}}{\pi\sqrt{n}}\}.$$

Formula (1) is an incredible evaluation of the celebrated Ramanujan (infinite) continued fraction at $e^{-2\pi/\sqrt{5}}$, and formulae (2) is related to the famous asymptotic series for the partition function that dominates the movie and will be discussed below.

The scene showing Hardy receiving a letter from an unknown Hindu clerk, and reading it in curiosity and amazement is really well done. Hardy's initial reaction was that Ramanujan was a fraud of a genius, but then he soon realizes in conversation with his distinguished colleague J. E. Littlewood, that Ramanujan is a genius ranking with the most illustrious mathematicians in history. The scenes showing Hardy in discussion with other Cambridge dons about the letter he has received from a Hindu clerk, and his decision to invite that clerk to Cambridge, are really impressive.

As soon as Ramanujan arrives in England, Hardy is ready to proceed in full swing with discussions about Ramanujan's mathematics. These scenes show effectively how Hardy is totally focused on work. Hardy being a sophisticated mathematician, naturally wanted proofs for Ramanujan's outrageous claims such as (1). Hardy had asked for proofs even when he responded to Ramanujan's letters, and in discussions in Cambridge, Hardy points out that Ramanujan's claims for formulas relating to prime numbers were actually false as stated. It is true that Ramanujan's one great failure as emphasized by Hardy were his assertions concerning the distribution of primes, but a vast majority of his results on infinite series, products, and integrals were actually correct, and Hardy realized that.

Even today it is not clear how Ramanujan's mind worked. Hardy being a sworn athiest, dismissed the Goddess of Namakkal story as mere fable. There is a scene in the movie when Hardy asks Ramanujan how he discovered his results. This actually never happened. In his famous twelve lectures on Ramanujan's work ([11], p. 11), Hardy admits that he was in the best position to clear up the mystery of how Ramanujan arrived at his results, but he blames himself for not even once asking Ramanujan about this. Hardy says that Ramanujan was showing him a dozen new theorems each day, with the result that his time was spent understanding these theorems, and so he never asked Ramanujan what his motivation or source of inspiration was.

As in the case of some other stage productions on Ramanujan, the Hardy-Ramanujan asymptotic series for the partition function is the focus of the discussion in INFINITY with regard to Ramanujan's work. Back in 2005, I saw a play about Ramanujan entitled *Partition*, which was centered around the amazing Hardy-Ramanujan formula for the partition function (see my review which appeared in The Hindu, India's National Newspaper in May 2005, but is reprinted in [2], pp. 143-44). There are several reasons why the Hardy-Ramanujan partition formula is the primary choice of all stage productions on Ramanujan including INFINITY: (i) Partitions are easy to explain and comprehend unlike other aspects of Ramanujan's work which require a background in mathematics to understand them. (ii) The Hardy-Ramanujan formula for partitions is one of the greatest achievements in number theory, and is stunning to anyone who sees the formula, and (iii) it shows how the brilliant intuition of Ramanujan and the sophistication of Hardy combined beautifully to discover and prove this remarkable result.

A partition of a positive integer n is a representation of n as a sum of positive integers, two such representations being considered the same if they differ only in the order of the summands (parts). For example, 2+2+1 is the same partition of 5 as 2+1+2. We denote the number of partitions of n by $p(n)$. There are five partitions of 4, namely 4, 3+1, 2+2, 2+1+1 and 1+1+1+1, and so $p(4) = 5$. Now in the case of a small number like 4, one could write down ALL its partitions and hence conclude that $p(4) = 5$. But $p(n)$ grows very rapidly and so it will not be easy or possible to write down ALL partitions of n in order to compute the value of $p(n)$. Take $n = 200$ for instance. We have

$$p(200) = 3972999029388 \tag{3}$$

which is about 4 trillion, and so it would be impossible to write down all partitions of 200. So how do we know the value of $p(200)$ without writing down all the partitions of 200? This is what mathematicians are very good at doing!

The great Leonard Euler, the founder of the theory of partitions in the mid-eighteenth century, discovered the following remarkable recurrence relation:

$$p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)-p(n-22)-p(n-26)+\cdots \tag{4}$$

This permits the calculation of $p(n)$ from the values of the partition function at smaller integers. For example $p(11) = p(10) + p(9) - p(6) - p(4)$. Arago said ([7], p. 139) that "Euler calculated without apparent effort as men breathe or as eagles sustain themselves in the wind". The numbers 1, 2, 5, 7, 12, 15, 22, 26, ... in the above recurrence are the Pentagonal numbers given by the formula $(3k^2 \pm k)/2$.

Even though (4) is remarkable both in appearance and in efficiency, it is not a closed form evaluation of $p(n)$. What Hardy and Ramanujan wanted, and successfully obtained, was a representation of $p(n)$ in terms of familiar continuous functions (like we encounter in calculus). This seemed impossible because partitions represent a discrete process, and so how can one expect a representation in terms of continuous functions? In the movie, this is what Hardy means by saying "Partitions can't be done". Hardy in true life would never have spoken like this. His statements were always measured and accurate. But the Director is reaching out to the millions here, not just to a mathematically literate audience. So by saying that "Partitions can't be done", the Director is conveying the near impossibility of such a representation using continuous functions without getting tangled in technical jargon that the the lay public would not comprehend.

Euler's interest was not just on $p(n)$ but on other partition functions as well, because many partition functions have generating functions which have elegant infinite product or series representations; these generating function evaluations yield beautiful relations among various partition functions. His famous formula for the generating function of $p(n)$ is

$$F(z) := \sum_{n=0}^{\infty} p(n) z^n = \prod_{m=1}^{\infty} \frac{1}{(1-z^m)}, \quad \text{for} \quad |z| < 1. \tag{5}$$

From the fundamental Cauchy Residue Theorem in complex variable theory (which came a few decades after Euler's time), it follows that for each positive integer n

$$p(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{F(z)}{z^{n+1}} dz, \tag{6}$$

where i is the imaginary square root of -1, and the integral is be taken counter clockwise over a simple closed contour $\mathcal{C}$ encircling the origin and within the unit circle. The difficulty is to determine the contour that would lead to the evaluation of $p(n)$. Notice that the product part in (5) indicates that $F(z)$ would get large when q is near a root of unity. So the brilliant idea of Hardy-Ramanujan was to start with a simple circular contour and then to deform it to take it close the roots of unity to evaluate $p(n)$. This is very deep, difficult, and sophisticated, and is the heart of the powerful *circle method* that they introduced in their famous paper of 1918. The final result they prove is that

$$|p(n) - \frac{1}{2\sqrt{2}} \sum_{q=1}^{\nu} \sqrt{q} A_q(n) \psi_q(n)| = O(\frac{1}{n^{1/4}}), \tag{7}$$

where ν is of the order of magnitude $\sqrt{n}$, the $A_q(n)$ being sums over certain q-th roots of unity, $O(.)$ means less than a constant times whatever is within the parenthesis, and

$$\psi_q(n) = \frac{d}{dn}(\exp\{\pi \sqrt{\frac{2}{3}(n - \frac{1}{24})}/q\}). \tag{8}$$

What (7) is saying is that since the error term $O(n^{-1/4})$ is tending to zero as $n \to \infty$, the value of $p(n)$ is the nearest integer to the remarkable sum of continuous functions up to ν.

Just as Hardy says initially in the movie that "partitions can't be done", after (7) was established, he informs many of his Cambridge colleagues that "he has done it!" This definitely conveys the impression that Ramanujan was solely responsible for (7). Yes, (7) could not have been accomplished without Ramanujan's insight and insistence, but Hardy's technical prowess in complex variables was just as important in the proof. I will now explain what the contributions of Ramanujan and Hardy were in obtaining (7).

In mathematics there is a wonderful convention that if a joint paper is written, then ALL authors are equal, and so one should not dig into who did what in a collaboration in the interest of preserving the harmony of the joint effort. Hardy was a firm believer in this practice and therefore did not elaborate on the respective contributions of Ramanujan and himself in deriving (7). Fortunately Littlewood in reviewing Ramanujan's Collected Papers had (after securing Hardy's consent) discussed the individual contributions of Hardy and Ramanujan to (7), and so I have relied on Littlewood's account in [15].

All along, Ramanujan had insisted that it should be possible to obtain an accurate formula involving continuous functions that would yield the value of $p(n)$ but Hardy initially did not believe that. With n in the place of $n - \frac{1}{24}$, the asymptotic size of $p(n)$ was first determined. It was Ramanujan's insight to use $n - \frac{1}{24}$ and with this it was shown that by summing the series up to a fixed number of terms, the error was at most of the order of magnitude of the next term, which was appreciably smaller. Thus (7) gives an asymptotic series for $p(n)$. Ramanujan also had guessed the functions $A_q(n)$ and $\psi_q(n)$ and continued to insist that much more was true, and there ought to be a representation that gives the value of $p(n)$ with a bounded error. At this point Hardy asked P. A. MacMahon, a noted combinatorialist (more about him below), to check the formula for numerical accuracy. MacMahon calculated the value $p(200)$ in (3) using (4) and his computations showed the astonishing accuracy of (7) even with a few terms of the series. Following this, using complex variables Hardy could show as surmised by Ramanujan that if the sum in (7) is taken up to an order of magnitude of $\sqrt{n}$ terms, then one does get $p(n)$ as the nearest integer to the sum. The determination of the precise form of $\psi_q(n)$ was crucial, and Littlewood credits Ramanujan's brilliant intuition for this: "The form of the function $\psi_q(n)$ is a kind of indivisible unit; among the many asymptotically equivalent forms, it is essential to select exactly the right one... the $-\frac{1}{24}$ (to say nothing of d/dn) is an extraordinary stroke of formal genius, the complete result can never come into the picture at all. .... But why was Ramanujan so certain ... There seems no escape from the conclusion that the discovery of the correct form has a single stroke of (Ramanujan's) insight. We owe the theorem to a singularly happy collaboration of two men, of quite unlike gifts, in which each contributed the best, most characteristic, and most fortunate work that was in him."

One of the peculiar properties of (7) is that if ν is replaced by ∞, the sum would diverge! This was established by Lehmer [14] in 1937. That same year, Hans Rademacher [16] noticed and proved that if the exponential function used in the definition of $\psi_q(n)$ is replaced by a suitable hyperbolic function, then this would convert the series in (7) to an infinite convergent series, whose value would yield $p(n)$! Although Ramanujan did not write down the Rademacher series, he felt intuitively that an something like that ought to exist. But Hardy felt that a convergent infinite series for $p(n)$ was too good to be true, and so he settled for less, namely the asymptotic series for $p(n)$, which itself was startling. In formula (2) above in Ramanujan's letter to Hardy, the expression of the right is equal to

$$\prod_{m=1}^{\infty} \frac{(1+q^m)}{(1-q^m)}, \tag{9}$$

which is the generating function for partitions where each partition is counted with weight

2 to the power of the number of different parts in the partition. Thus (2) and (9) are generating functions of a weighted version of $p(n)$. It is to be noted that in (2) Ramanujan was using hyperbolic functions just like Rademacher did for $p(n)$. While (2) is not correct as Ramanujan stated in his first letter to Hardy, it provides a genuine approximation. Hardy said ([9], p. 9) that "Ramanujan's false statement was one of the most fruitful he ever made since it led us to all our joint work on partitions". Probably because (2) was false as it stood, and because the use of the exponential function is slighly simpler compared to the hyperbolic functions, Hardy preferred the form in (7). In a thought-provoking lecture in Chennai, India, on December 22, 1987, Ramanujan's 100-th birthday (which I happened to attend), the late Fields Medalist Atle Selberg of the Institute for Advanced Study, Princeton, said that the Rademacher convergent series involving hyperbolic functions was somewhat more natural than the asymptotic series of Hardy-Ramanujan. Selberg said that he had independently obtained such a convergent series but did not publish it because he realized that Rademacher had done it earlier. Selberg pointed out that although Ramanujan felt that such a convergent series ought to exist, out of respect for his mentor Hardy, he (Ramanujan) agreed to stop with the asymptotic series for $p(n)$. Selberg emphasized that if Hardy had trusted Ramanujan's insight totally, they would have arrived at Rademacher's convergent series for $p(n)$. Selberg's comments can be found in [23] which is a write-up of another talk he gave a few weeks later in Bombay.

Hardy approached MacMahon to check the accuracy of the asymptotic formula for $p(n)$ because MacMahon was a wizard in computation. Hardy ([11], p. 119) credits MacMahon's calculations as being crucial for the final version of his result (7) with Ramanujan. There is a charming scene in the movie in which Hardy brings Ramanujan to MacMahon's office to introduce the Indian genius to him. In that scene, MacMahon challenges Ramanujan to compute the square root of a certain number which Ramanujan does instantly. In return, MacMahon asks Ramanujan to give him a computational problem and MacMahon gives the answer just as quickly. MacMahon was one of the fastest in numerical computation as Hardy himself has pointed out([20], p. xxxv and [11] p. 119), and there were friendly computational contests between MacMahon and Ramanujan from time to time; Hardy felt ([20], p. xxxv) that MacMahon was the faster and more accurate of the two. MacMahon, who was a former Major in the British Army, was stationed in India and even in Madras. He was a Fellow of St. John's College of Cambridge University but never a Cambridge faculty member. The movie shows MacMahon as somewhat conceited and Hardy seeking his approval in certain matters. In any case this scene showing the encounter between MacMahon and Ramanujan is quite humorous and projects effectively the computational prowess of both.

Hardy used the remarkable formula for partitions to get Ramanujan elected Fellow of the Royal Society (FRS) and Fellow of Trinity College. Even though Ramanujan had several spectacular results, Hardy needed something totally unexpected and staggering like the asymptotic series for $p(n)$ to get these recognitions for Ramanujan. But these did not come by easily. The scenes showing Hardy's tremendous efforts in trying to convince the British academic aristocracy are moving. Hardy's first attempt to get Ramanujan elected Fellow of Trinity was unsuccessful. Subsequently he succeeded in getting Ramanujan elected Fellow of the Royal Society, following which the Fellowship of Trinity came without resistance.

In connection with the Fellowship of Trinity College, there are a couple of humorous scenes: Soon after Ramanujan arrives in Cambridge, one day while he is on his way to meet Hardy, he was about to walk across the lawn at Trinity College. Immediately the guard stops Ramanujan and orders him to walk along the path surrounding the lawn. To the British, preserving the lawn is sacrosanct, and only the priviledged like Fellows of the College can tread on those immaculate lawns! After Ramanujan is elected Fellow of Trinity, Hardy tells him that now he could walk across the lawn with confidence! The producers and the Director have to be applauded that so many interesting aspects of Ramanujan's life in England have been tastefully presented within the short time span of the movie.

The circle method invented by Hardy and Ramanujan to get the asymptotic series for $p(n)$ is indeed one of the most important and powerful methods in number theory, and therefore deserves the pride of place in any stage production on Ramanujan such as INFINITY, or in any exposition of Ramanujan's contributions. In a series of five papers under the title "Some problems in Partition Numerorum", Hardy in collaboration with Littlewood extended the circle method to make it the principal tool for a wide class of problems in Additive Number Theory, namely the part of number theory that deals with additive questions such as the number of ways in which an integer can be represented as a sum of a certain number of primes, or k-th powers, etc.

Besides the asymptotic formula for $p(n)$, Ramanujan obtained many other deep and unexpected results on partitions; indeed the thory of partitions underwent a glorious transformation owing to his magic touch! Many of Ramanujan's identities involving q-hypergeometric series have significant partition implications. Andrews' comprehensive book [4] includes a discussion of Ramanujan's broad range of results on partitions and q-hypergeometric series and the developments stemming from them.

While Ramanujan had a lot of respect for Hardy and did not pursue the convergent series for $p(n)$ in deference to Hardy, he was actually quite confident of the correctness of his results in general, and therefore was not as timid in the presence of Hardy as depicted in the movie. Similarly, while Hardy did insist on proofs and conveyed to Ramanujan that proofs are central to mathematics, he did not chastise Ramanujan to the extent as shown in the movie where in one scene Hardy tells Ramanujan that he would not see him again unless he brought with him proofs of certain results. Hardy was never so harsh on Ramanujan. In his Preface to Ramanujan's Collected Papers, Hardy says ([20], xxx-xxxi): "It is impossible to ask such a man to submit to systematic instruction...I was afraid also that, if I insisted unduly on matters which Ramanujan found irksome, I might destroy his confidence or break the spell of his inspiration...He was never a mathematician of the modern school, and it was hardly desirable he should become one; but he knew when he had proved a theorem and when he had not. And his flow of original ideas shewed no symptom of abatement". In reality Hardy helped Ramanujan write up his results, filled some missing steps in the arguments, so that Ramanujan's wonderful work could be published. One scene shows Ramanujan jumping with joy and surprise when Hardy hands him a reprint of his paper on highly composite numbers [19] that had just appeared in the Proceedings of the London Mathematical Society. The Director has dramatized the contrast between Hardy emphasizing proofs and Ramanujan's insight in writing down formulas without proofs, to evince audience interest, and this is understandable. But the dramatizations relating to discrimination or racial prejudice against Ramanujan in England is overdone in the movie, because certain characters are tainted in the presentation. I discuss two such scenes here.

At Hardy's suggestion, Ramanujan attended some lectures to get some basic knowledge of some important areas of mathematics. One scene shows Ramanujan with other students in a class in which the lecturer is discussing certain formulas involving special functions. The lecturer observes that unlike other students in the class, Ramanujan is not taking notes. So the lecturer asks Ramanujan if he is following the discussion, to which Ramanujan replies that he is, and that he knows the answer. Somewhat irritated by this reply, the lecturer asks Ramanujan to come to the board and demonstrate his solution, which Ramanujan does with absolute ease and with astonishing speed. The infuriated lecturer throws Ramanujan out of the class and asks Ramanujan not to attend any more lectures. While this scene effectively conveys Ramanujan's brilliance and the way in which he often startled his professors with his mathematical prowess, in reality they actually admired his genius and did not admonish him as depicted in this scene. To describe what really happened, I quote P. C. Mahalanobis, a collegemate of Ramanujan in Cambridge, who later founded the famous Indian Statistical Institute in Calcutta: *I joined King's College in Cambridge in October 1913. I was attending some mathematical courses at that time including one by Professor Hardy. A little later, we heard that S. Ramanujan, the mathematical prodigy, would come to Cambridge. I used to do my tutorial work with Mr. Arthur Berry, Tutor in Mathematics of King's College. One day I was waiting in his room for my tutorial when he came in after having taken a class on elliptic integrals. He asked me: "Have you met your wonderful countryman, Ramanujan?" I told him I had heard that he had arrived but that I had not met him so far. Mr. Berry said: "He came to my elliptic integrals class this morning".... I asked "What happened? Did he follow your lecture?" Mr. Berry said, "I was working out some formulae on the black board. I was looking at Ramanujan from time to time to see if he was following what I was doing. At one stage, Ramanujan's face was beaming and he appeared to be excited. I asked him whether he was following the lecture and Ramanujan nodded his head. I then enquired whether he would like to say anything. He got up from this seat, went to the black board and wrote some of the results which I had not yet proved."*

I remember that Mr. Berry was greatly impressed. He said that Ramanujan must have reached those results by pure intuition and Professor Hardy had advised him to attend the lectures on elliptic integrals because Ramanujan had not studied that subject before.

This quote of Mahalanobis is taken from P. K. Srinivasan's wonderful book ([25], pp. 145-148). Kanigel describes this story without referring to Mahalanobis (see [13], pp.201-202), but he does acknowledge Srinivasan's book in describing this incident.

There are several instances in rural schools and colleges in India where lecturers have admonished students for demonstrating their superior skills, but such admonishment would never happen in such hallowed centres of learning like Cambridge. Although this dramatization projects the brillance of Ramanujan, it does so at the cost of tainting the image of a respectable Cambridge faculty member, who in the movie is not Berry.

In another scene, Ramanujan goes to the post office to see if there are any letters for him from his wife to be picked up; he had not heard from her because in reality his mother had hidden those letters and not posted them to Ramanujan. When Ramanujan leaves the post office totally dejected, a group of young white lads in military uniform, make fun of Ramanujan, kick him and push him to the ground, and Ramanujan's face is bloodied. This never happened. There was never a single instance of physical abuse of Ramanujan in England due to racial prejudice, and so this scene maligns the British security forces unnecessarily.

No account of Ramanujan is complete without the famous taxi cab episode. One day Hardy was visiting Ramanujan who was in a nursing home in Putney. In order to make light conversation, Hardy says he must have ridden in a taxi with a rather dull number to see Ramanujan in such a sorry state. When Ramanujan asked what the number was, Hardy said it was 1729. Immediately Ramanujan exclaimed that this is not a dull number but a very interesting one, because 1729 is the smallest number which can be written as a sum of two cubes in two different ways:

$$1729 = 1728 + 1 = 12^3 + 1^3 = 1000 + 729 = 10^3 + 9^3. \tag{10}$$

Hardy was stunned that Ramanujan could immediately come up with this remarkable property of 1729. It was such incidents that prompted Littlewood to say that every number was a personal friend of Ramanujan! Actually, back in India, Ramanujan had worked out the parametrization of the all the integer solutions to the equation

$$x^3 + y^3 = z^3 + w^3 \tag{11}$$

and knew that 1729 was the smallest solution with x, y, z, w all positive integers. It was a sheer coincidence that Hardy arrived in the taxi numbered 1729, but I would prefer to think of it as the action of the Goddess of Namakkal! Prior to Ramanujan, Euler had obtained the parametrization of all the solutions to (11), and so it is called Euler's equation, but Ramanujan's parametrization is more elegant (see [9]).

In the movie there are several scenes showing discussion between Hardy and Ramanujan when Ramanujan was lying ill in hospitals in England. These scenes effectively show that in spite of poor health, Ramanujan was obsessed with his mathematical research. In all these hospital scenes, the mathematical discussion is about partitions and never about 1729. The 1729 taxi cab episode is brought in near the end of the movie when Ramanujan takes a taxi to the port to sail back to India, and lovely property (10) of 1729 mentioned by Ramanujan to Hardy eases the emotion of the parting scene, which is touching and beautifully done in the movie. While the general public outside India may not know how and where the 1729 episode occurred, the billion people of India know this story very well, and would note this discrepancy in the movie instantly.

The movie concludes with Hardy's emotional speech to the Royal Society announcing Ramanujan's demise in India. Very prudently, the Director had chosen not to show Ramanujan's final few months in India where he suffered a lot. After Ramanujan died, most of his relatives boycotted his cremation since they felt he had sinned by crossing the oceans. If the Director had elected to also include Ramanujan's final months in India and his demise there, the worldwide public would have been drawn into a discussion of how poorly and unfairly Ramanujan was treated by his orthodox relatives; this would have distracted the audience instead of focussing on the great achievements of Ramanujan. So once again I should say that the Director wisely eschewed the scenes of the final months of Ramanujan in India, just as he did not allude to Ramanujan's marriage to Janaki when she was only a child.

Fostering the legacy of Ramanujan: It is now nearly a century after Ramanujan passed away, but his mathematical influence is continuing to grow. His legacy is being fostered in various ways - books explaining his findings recorded in his Notebooks and the Lost Notebook and making comparisions with contemporary research have been written [6], [8], a journal devoted to all areas of mathematics influenced by him was launched in 1997 and has tripled in size since then (see [2], pp. 147-151), prizes for very young mathematicians for outstanding research in areas influenced by Ramanujan have been created (see [2], pp. 161-166) - Manjul Bhargava, one of the associate producers of the movie, won this prize in 2005 the very first year it was awarded, research is being conducted in leading centers around the world on mathematical topics stemming directly from Ramanujan's discoveries, popular books on Ramanujan's life and mathematics have been written, stage productions like plays, documentaries, movies, and even an opera about his life and work have been produced; I have described various efforts to foster his legacy in an article to the American Mathematical Society (see [3]). But among all these, Kanigel's book has had the widest appeal to the worldwide public because it is a compelling and comprehensive account of the fascinating life of Ramanujan. Now this movie based on Kanigel's book brings that story to the silver screen to educate and inspire viewers around the world. There is not a single dull moment in the movie and my attention was riveted to the screen all along. Although I have pointed out discrepancies for the sake of clarity, I wish to reiterate that this movie, excluding documentaries, is one of the finest among all movies on mathematics ever produced. In the Preface to the first issue of The Ramanujan Journal, I said "The very mention of Ramanujan's name reminds us of the thrill of mathematical discovery". The movie INFINITY not only conveys the thrill of such a discovery effectively, but it also movingly and tastefully depicts the remarkable life of Srinivasa Ramanujan - a truly exceptional figure in human history.

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