9. (Richards 1982) For every prime $p \le y$, let $\beta(p)$ denote the greatest positive integer such that $p^\beta \le y$, and put

$$q = \prod_{\substack{p \le y \\ p \equiv 3\,(4)}} p^{2\beta(p)}.$$

(a) Show that $q = \exp(2\psi(y; 4, 3))$.

(b) Show that $\log q \ll y$.

(c) Suppose that $1 \le n \le y$. Show that if $n \equiv 3 \pmod 4$, then there is a prime $p|q$ such that $p$ divides $n$ to an odd power.

(d) Let $x = (q-1)/4$. Show that $x$ is an integer, and that $4x \equiv -1 \pmod q$.

(e) Show that if $1 \le i \le y/4$ and $p|q$, then the power of $p$ that exactly divides $x + i$ is the same as the power of $p$ that exactly divides $4i - 1$.

(f) Deduce that no integer in the interval $(x, x + y/4]$ can be expressed as a sum of two squares.

(g) Conclude that there exist arbitrarily large numbers $x$ such that no number between $x$ and $x + c\log x$ is a sum of two squares. Here $c$ is a suitably small positive constant.

## 7.4 Numbers composed of a prescribed number of primes

Let $\sigma_k(x)$ denote the number of integers $n$ with $1 \le n \le x$ and $\Omega(n) = k$. The $\sigma_1(x) = \pi(x) \sim x/\log x$. Consider $\sigma_2(x)$. Clearly

$$\sigma_2(x) = \sum_{\substack{p_1, p_2 \\ p_1 \le p_2 \\ p_1 p_2 \le x}} 1 = \sum_{p \le \sqrt{x}} \left(\pi(x/p) - \pi(p) + O(1)\right).$$

By the Prime Number Theorem this is

$$= \sum_{p \le \sqrt{x}} (1 + o(1)) \frac{x}{p(\log x/p)} + O\left(\frac{x}{\log x}\right).$$

Thus, by partial summation and a further application of the Prime Number Theorem we find that

$$\sigma_2(x) \sim \frac{x \log\log x}{\log x}.$$

By inducting on $k$ in this manner it can be shown that

$$\sigma_k(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\,\log x}$$

for any fixed $k$. Since the sum over all $k \geq 1$ of the right-hand side is exactly $x$, it is tempting to think that the above holds quite uniformly in $k$. However this is not the case, as we shall presently discover. To obtain precise estimates that are uniform in $k$ we apply analytic methods. In Section 2.4 we determined the asymptotic distribution of the additive function $\Omega(n) - \omega(n)$ by establishing the mean value of the multiplicative function $z^{\Omega(n)-\omega(n)}$. In the same spirit we shall derive information concerning the distribution of $\Omega(n)$ from mean value estimates of $z^{\Omega(n)}$. Since the Euler product of this latter function behaves badly when $|z|$ is large, we start not with $z^{\Omega(n)}$ but with $d_z(n)$ defined by the identities

$$\zeta(s)^z = \prod_p \left(1 - p^{-s}\right)^{-z} = \sum_{n=1}^{\infty} d_z(n) n^{-s} \qquad (\sigma > 1). \tag{7.56}$$

Since $d_z(p) = z = z^{\Omega(p)}$, the functions $d_z(n)$ and $z^{\Omega(n)}$ are 'nearby', and hence the mean value of $z^{\Omega(n)}$ can be derived from that for $d_z(n)$ by elementary reasoning.

**Theorem 7.17** *Let $D_z(x) = \sum_{n \leq x} d_z(n)$, and let $R$ be any positive real number. If $x \geq 2$, then*

$$D_z(x) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O\left(x(\log x)^{\Re z - 2}\right)$$

*uniformly for $|z| \leq R$.*

*Proof* Let $a = 1 + 1/\log x$. Then by Corollary 5.3,

$$\begin{aligned} D_z(x) - \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \zeta(s)^z \frac{x^s}{s}\, ds \ll &\sum_{\frac{1}{2}x < n < 2x} |d_z(n)| \min\left(1, \frac{x}{T|x-n|}\right) \\ &+ \frac{x^a}{T} \sum_n |d_z(n)| n^{-a}. \end{aligned} \tag{7.57}$$

Since $|d_z(n)|$ is erratic, we must exercise some care in estimating the error terms above. Let $\mathcal{A} = \{n : |n - x| \leq x/(\log x)^{2R+1}\}$. Without loss of generality we may suppose that $R$ is an integer. We note that $|d_z(n)| \leq d_{|z|}(n) \leq d_R(n)$. By the method of the hyperbola we see by induction on $R$ that

$$D_R(x) = x P_R(\log x) + O_R\left(x^{1-1/R}\right)$$

where $P_R$ is a polynomial of degree $R - 1$. Hence the contribution to the first sum in the error term in (7.57) of the $n \in \mathcal{A}$ is

$$\ll \sum_{n \in \mathcal{A}} |d_z(n)| \ll x(\log x)^{-R-2}$$

The contribution of the $n \notin \mathcal{A}$ is

$$\ll T^{-1}(\log x)^{2R+1}x(\log x)^{R-1}.$$

We take $T = \exp\left(\sqrt{\log x}\right)$ to see that this is also $\ll x(\log x)^{-R-2}$. The second sum in the error term in (7.57) is $\ll \zeta(a)^R \ll (\log x)^R$. Thus the total error term is $\ll x(\log x)^{-R-2}$.

If $z$ is a positive integer, then $\zeta(s)^z$ has a pole at $s = 1$, and we can extract a main term by the calculus of residues, as in our proof of the Prime Number Theorem (Theorem 6.9). On the other hand, if $z$ is not an integer, then $\zeta(s)^z$ has a branch point at $s = 1$, so greater care must be exercised in moving the path of integration. Put $b = 1 - c/\log T$ where $c$ is a small positive constant, and replace the contour from $a - iT$ to $a + iT$ by a path consisting of $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ where $\mathcal{C}_1$ is a polygonal with vertices $a - iT$, $b - iT$, $b - i/\log x$, $\mathcal{C}_2$ begins with a line segment from $b - i/\log x$ to $1 - i/\log x$, continues with the semicircle $\{1 + e^{i\theta}/\log x : -\pi/2 \le \theta \le \pi/2\}$, and concludes with the line segment from $1 + i/\log x$ to $b + i/\log x$, and finally $\mathcal{C}_3$ is polygonal with vertices $b + i/\log x, b + iT, a + iT$. By Theorem 6.7, $\zeta(s)^z \ll (\log x)^R$ on the new path, so the integrals over $\mathcal{C}_1$ and $\mathcal{C}_3$ contribute an amount $\ll x(\log x)^{-R-2}$. On $\mathcal{C}_2$ we have $\zeta(s)^z/s = (s-1)^{-z}(1 + O(|s-1|))$. Hence

$$\frac{1}{2\pi i}\int_{\mathcal{C}_2}\zeta(s)^z\frac{x^s}{s}\,ds = \frac{1}{2\pi i}\int_{\mathcal{C}_2}(s-1)^{-z}x^s\,ds + O\left(\int_{\mathcal{C}_2}|s-1|^{1-\Re z}x^\sigma\,|ds|\right). \tag{7.58}$$

By the change of variables $s = 1 + w/\log x$ we see that the main term above is

$$x(\log x)^{z-1}\frac{1}{2\pi i}\int_{\mathcal{H}_2} w^{-z}e^w\,dw$$

where $\mathcal{H}_2$ starts at $-\beta - i$, loops around 0, and ends at $-\beta + i$ where $\beta = c(\log x)/\log T$. Let $\mathcal{H}_1$ be the contour $\mathcal{H}_1 = \{w = u - i : -\infty < u \le -\beta\}$, and similarly let $\mathcal{H}_3 = \{w = u + i : -\infty < u \le -\beta\}$. If we integrate over the union of the $\mathcal{H}_i$, then we obtain Hankel's formula (see Theorem C.3) for $1/\Gamma(z)$. The integral over $\mathcal{H}_1$ is $\ll_R \int_\beta^\infty e^{-u/2}\,du \ll_R e^{-\beta/2}$, which is small since $T = \exp(\sqrt{\log x})$. Thus we see that the main term in (7.58) is $x(\log x)^{z-1}/\Gamma(z) + O_R(x\exp(-c\sqrt{\log x}))$ for some constant $c$. On the semicircular part of $\mathcal{C}_2$ the integrand in the error term in (7.58) is $\ll x(\log x)^{\Re z - 1}$, so the contribution is $\ll x(\log x)^{\Re z-2}$. By the change of variables $s = 1 + w/\log x$ we see that the linear portions of $\mathcal{C}_2$ contribute an amount

$$\ll x(\log x)^{\Re z-2}\int_0^\infty (u^2+1)^{(R-1)/2}e^{-u}\,du \ll_R x(\log x)^{\Re z-2}.$$

Thus we have the stated estimate, and the proof is complete. □

We now establish a procedure by which we can pass from $d_z(n)$ to other nearby functions.

**Theorem 7.18** *Suppose that* $\sum_{m=1}^{\infty} |b_z(m)|(\log m)^{2R+1}/m$ *is uniformly bounded for* $|z| \le R$, *and for* $\sigma \ge 1$ *let*

$$F(s,z) = \sum_{m=1}^{\infty} b_z(m) m^{-s}.$$

*Let* $a_z(n)$ *be defined by the relation*

$$\zeta(s)^z F(s,z) = \sum_{n=1}^{\infty} a_z(n) n^{-s} \qquad (\sigma > 1)$$

*and let* $A_z(x) = \sum_{n \le x} a_z(n)$. *Then for* $x \ge 2$,

$$A_z(x) = \frac{F(1,z)}{\Gamma(z)} x (\log x)^{z-1} + O\big(x (\log x)^{\Re z - 2}\big).$$

*Proof* Since $a_z(n) = \sum_{m|n} b_z(m) d_z(n/m)$, we see by Theorem 7.17 that

$$\begin{aligned} A_z(x) &= \sum_{m \le x/2} b_z(m) D_z(x/m) + \sum_{x/2 < m \le x} b_z(m) \\ &= \frac{x}{\Gamma(z)} \sum_{m \le x/2} \frac{b_z(m)}{m} (\log x/m)^{z-1} + O\left( x \sum_{m \le x} \frac{|b_z(m)|}{m} (\log 2x/m)^{\Re z - 2} \right). \end{aligned} \tag{7.59}$$

The error term here is

$$\begin{aligned} &\ll x(\log x)^{\Re z - 2} \sum_{m \le \sqrt{x}} \frac{|b_z(m)|}{m} + x(\log x)^{-R-2} \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m} (\log m)^{2R} \\ &\ll x(\log x)^{\Re z - 2}. \end{aligned}$$

In the main term, when $m \le x^{1/2}$ we write

$$(\log x/m)^{z-1} = (\log x)^{z-1} + O\big((\log m)(\log x)^{\Re z - 2}\big).$$

Thus the first sum on the right-hand side of (7.59) is

$$\begin{aligned} &= (\log x)^{z-1} \sum_{m \le x/2} \frac{b_z(m)}{m} \\ &\quad + O\left( (\log x)^{\Re z - 2} \sum_{m \le \sqrt{x}} \frac{|b_z(m)|}{m} \log m + (\log x)^{R-1} \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m} \right) \\ &= (\log x)^{z-1} F(1,z) + O\left( (\log x)^{\Re z - 2} \sum_{m} \frac{|b_z(m)|}{m} (\log m)^{2R+1} \right), \end{aligned}$$

which gives the result. □

Suppose that $R < 2$, and let

$$F(s,z) = \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z \tag{7.60}$$

for $\sigma > 1$, $|z| \le R$. Then $a_z(n) = z^{\Omega(n)}$ in the notation of Theorem 7.18. Hence, with $\sigma_k(x)$ defined as at the beginning of this section we find that

$$A_z(x) = \sum_{n \le x} z^{\Omega(n)} = \sum_{k=0}^{\infty} \sigma_k(x) z^k.$$

Here the power series on the right is actually a polynomial, since $\sigma_k(x) = 0$ for sufficiently large $k$, when $x$ is fixed. Our asymptotic estimate for $A_z(x)$ enables us to recover an estimate for the power series coefficients $\sigma_k(x)$, since Cauchy's formula asserts that

$$\sigma_k(x) = \frac{1}{2\pi i} \int_{|z|=r} \frac{A_z(x)}{z^{k+1}}\, dz \tag{7.61}$$

for $r < 2$.

**Theorem 7.19** *Suppose that $R < 2$, that $F(s,z)$ is given by (7.60), and that $G(z) = F(1,z)/\Gamma(z+1)$. Then*

$$\sigma_k(x) = G\left(\frac{k-1}{\log\log x}\right) \frac{x(\log\log x)^{k-1}}{(k-1)!\,\log x} \left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right) \tag{7.62}$$

*uniformly for $1 \le k \le R \log\log x$.*

Since $G(0) = G(1) = 1$, we see that (7.55) holds when $k = o(\log\log x)$, and also when $k = (1 + o(1)) \log\log x$, but that (7.55) does not hold in general. The restriction to $R < 2$ is necessary because of the contribution of the prime $p = 2$ in the Euler product (7.60) for $F(s,z)$. If $z \ge 2$, then the behaviour is different; see Exercises 7.4.5 and 7.4.6, below.

*Proof* Our quantitative form of the Prime Number Theorem (Theorem 6.9) gives the case $k = 1$, so we may assume that $k > 1$. We substitute the estimate of Theorem 7.18 in (7.61) with $r = (k-1)/\log\log x$. The error term contributes an amount

$$\ll x(\log x)^{r-2} r^{-k} = \frac{x}{(\log x)^2} e^{k-1} \frac{(\log\log x)^k}{(k-1)^k}$$
$$\ll \frac{x(\log\log x)^k}{(k-1)!(\log x)^2} \ll \frac{x(\log\log x)^{k-3}}{(k-1)!\,\log x}.$$

This is majorized by the error term in (7.62) since $G((k-1)/\log\log x) \gg 1$. The main term we obtain from (7.61) is $xI/\log x$ where

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{|z|=r} G(z)(\log x)^z z^{-k}\,dz \\ &= \frac{G(r)}{2\pi i}\int_{|z|=r} (\log x)^z z^{-k}\,dz + \frac{1}{2\pi i}\int_{|z|=r} (G(z)-G(r))(\log x)^z z^{-k}\,dz. \end{aligned}$$

By integration by parts we find that

$$\frac{r}{2\pi i}\int_{|z|=r} (\log x)^z z^{-k}\,dz = \frac{1}{2\pi i}\int_{|z|=r} (\log x)^z z^{1-k}\,dz.$$

We multiply both sides by $G'(r)$ and combine with the former identity to see that

$$\begin{aligned} I &= \frac{G(r)}{2\pi i}\int_{|z|=r} (\log x)^z z^{-k}\,dx \\ &\quad + \frac{1}{2\pi i}\int_{|z|=r} (G(z)-G(r)-G'(r)(z-r))(\log x)^z z^{-k}\,dz. \end{aligned} \tag{7.63}$$

Here the first integral is $(\log\log x)^{k-1}/(k-1)!$ by Cauchy's theorem, which gives the desired main term. On the other hand,

$$G(z)-G(r)-G'(r)(z-r) = \int_r^z (z-w)G''(w)\,dw \ll |z-r|^2,$$

so that if we write $z = re^{2\pi i\theta}$, then the second integral in (7.63) is

$$\ll r^{3-k}\int_{-1/2}^{1/2} (\sin\pi\theta)^2 e^{(k-1)\cos 2\pi\theta}\,d\theta.$$

But $|\sin x| \le |x|$ and $\cos 2\pi\theta \le 1 - 8\theta^2$ for $-1/2 \le \theta \le 1/2$, so the above is

$$\begin{aligned} &\ll r^{3-k}e^{k-1}\int_0^\infty \theta^2 e^{-8(k-1)\theta^2}\,d\theta \ll r^{3-k}e^{k-1}(k-1)^{-3/2} = \frac{(\log\log x)^{k-3}e^{k-1}}{(k-1)^{k-3/2}} \\ &\ll k(\log\log x)^{k-3}/(k-1)!. \end{aligned}$$

This completes the proof of the theorem. □

The decomposition in (7.63) is motivated by the observation that $|(\log x)^z|$ is largest, for $|z| = r$, when $z = r$. We take the Taylor expansion to the second term because

$$\left|\int (z-r)^2(\log x)^z z^{-k}\,dz\right| \asymp \int |z-r|^2|(\log x)^z z^{-k}|\,|dz|,$$

whereas

$$\left|\int (z-r)(\log x)^z z^{-k}\,dz\right| = o\left(\int |z-r||(\log x)^z z^{-k}|\,|dz|\right).$$

By the calculus of residues we may write

$$I = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left(G(z)(\log x)^z\right)\Big|_{z=0}$$
$$= \sum_{\nu=0}^{k-1}\frac{G^{(\nu)}(0)}{\nu!}\frac{(\log\log x)^{k-1-\nu}}{(k-1-\nu)!}.$$

This gives a more accurate, but more complicated, main term.

In Section 2.3 we saw that $\Omega(n)$ rarely differs very much from $\log\log n$. In particular, from Theorem 2.12 we see that if $r < 1$, then the number of $n \le x$ for which $\Omega(n) < r\log\log n$ is $\ll_r x/\log\log x$. We now give a much sharper upper bound for the number of occurrences of such large deviations.

**Theorem 7.20** *Let $A(x, r)$ denote the number of $n \le x$ such that $\Omega(n) \le r\log\log x$, and let $B(x, r)$ denote the number of $n \le x$ for which $\Omega(n) \ge r\log\log x$. If $0 < r \le 1$ and $x \ge 2$, then*

$$A(x, r) \ll x(\log x)^{r-1-r\log r}.$$

*If $1 \le r \le R < 2$ and $x \ge 2$, then*

$$B(x, r) \ll_R x(\log x)^{r-1-r\log r}.$$

*Proof* We argue directly from Theorem 7.18, using a modified form of Rankin's method. If $0 \le r \le 1$ and $\Omega(n) \le r\log\log x$, then $r^{r\log\log x} \le r^{\Omega(n)}$. Hence

$$A(x, r) \le (\log x)^{-r\log r}\sum_{n\le x} r^{\Omega(n)}.$$

By Theorem 7.18 this is

$$\sim \frac{F(1, r)}{\Gamma(r)}x(\log x)^{r-1-r\log r}$$

where $F(s, z)$ is taken as in (7.60). This gives the result since $F(1, r) \ll 1$ and $\Gamma(r) \gg 1$ uniformly for $0 < r \le 1$.

Now suppose that $1 \le r \le R < 2$ and that $\Omega(n) \ge r\log\log x$. Then $r^{\Omega(n)} \ge r^{r\log\log x}$, and hence

$$B(x, r) \le (\log x)^{-r\log r}\sum_{n\le x} r^{\Omega(n)}.$$

Thus we have only to proceed as before to obtain the result. □