NOTE ON A PAPER BY L. G. SATHE

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...e the results of Sathe's paper [*J. Indian Math. Soc.* 17 (1953), ...41; 18 (1954), 27-81] are very beautiful and highly interesting, ...ay the author has proceeded in order to prove these results ...rather complicated and involved one, and this by necessity ... a proof by induction after ν starting from the case $\nu = 1$, ...nts overwhelming difficulties in keeping track of the estimates ...the remainder terms in their dependence of the two parameters ...d x.

...simpler and more natural way to obtain these results will be ...ched briefly below. This approach also leads to a greater range ...validity for some of Sathe's results. The natural starting point ...the general divisor problem concerned with estimating the ...ression $\sum_{n \leqslant x} d_z(n)$, where

$$\sum_{n=1}^{\infty} \frac{d_z(n)}{n^s} = \{\zeta(s)\}^z,\ d_z(1) = 1, \tag{1}$$

...$s = \sigma + it$, $\sigma > 1$. $\zeta(s)$ here is Riemann's Zeta function. The ...tion of this problem is well known if z is a rational integer, but ...ms never to have been stated in the literature for arbitrary ...plex z, though it would seem likely to have been known to ...ialists in the field, since the proof is so straightforward along ...ical lines. From this result, our Theorem 1, the rest follows ...hout any difficulty.

In the following A is an arbitrary positive constant, and B a ...tive constant depending on the particular function $f(s, z)$, to ...r later. The constants involved in the O's depend only on A ...the case of Theorem 1, and only on the function $f(s, z)$ for the ...er theorems.

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THEOREM 1. *We have*

$$D_z(x) = \sum_{n \leqslant x} d_z(n) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O\{(x \log x)^{z-2}\},$$

uniformly for $|z| \leqslant A$, $x \geqslant 2$.

The proof proceeds along the standard lines, starting from the expression

$$\int_0^x D_z(t)\, dt = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)} \{\zeta(s)\}^z\, ds.$$

Deforming the path of integration using well-known estimates for $\log \zeta(s)$ in the neighbourhood of $\sigma = 1$, one obtains easily

$$\int_0^x D_z(t)\, dt = \frac{1}{2\pi i} \int_L \frac{x^{s+1}}{s(s+1)} \{\zeta(s)\}^z\, ds + O\left(\frac{x^2}{(\log x)^{3A+4}}\right), \qquad (2)$$

where the path of integration consists of the line segment $\{\frac{1}{2} - i/(\log x),\ 1 - i/(\log x)\}$ the halfcircle with center at 1 joining $1 - i/(\log x)$ and $1 + i/(\log x)$ to the right of the line $\sigma = 1$ and the line segment $\{1 + i/(\log x),\ \frac{1}{2} + i/(\log x)\}$. Using the fact that for $y \geqslant x \geqslant 2$,

$$D_z(y) - D_z(x) = O\{(y - x)(\log y)^{|z|}\} + O(y^{1-1/(1+|z|)}),$$

which follows easily from the fact that $|d_z(n)| \leqslant d_k(n)$, where $k = [|z|] + 1$, one gets from (2),

$$D_z(x) = \frac{1}{2\pi i} \int_L \frac{x^s}{s} \{\zeta(s)\}^z\, ds + O\left(\frac{x}{(\log x)^{A+2}}\right)$$

$$= \frac{1}{2\pi i} \int_L \frac{x^s}{(s-1)^z} \frac{1}{s} \{(s-1)\,\zeta(s)\}^z\, ds + O\left(\frac{x}{(\log x)^{A+1}}\right)$$

$$= \frac{x(\log x)^{z-1}}{\Gamma(z)} + O\{x(\log x)^{z-2}\},$$

where the last equation follows easily from the regularity of $(s-1)\,\zeta(s)$ around $s = 1$, and the fact that the value at $s = 1$ is 1.

THEOREM 2. *Let*

$$f(s, z) = \sum_{n=1}^{\infty} \frac{b_z(n)}{n^s}, \quad \textit{for } \sigma > 1,$$

and let $\sum_{n=1}^{\infty} |b_z(n)| n^{-1} (\log 2n)^{B+3}$ *be uniformly bounded for* $|z| \leqslant B$..
Further put

$$\{\zeta(s)\}^z f(s,z) = \sum_{n=1}^{\infty} \frac{a_z(n)}{n^s}, \sigma > 1.$$

Then, we have

$$A_z(x) = \sum_{n \leqslant x} a_z(n) = \frac{f(1,z)}{\Gamma(z)} x(\log x)^{z-1} + O\{x(\log x)^{z-2}\},$$

uniformly for $|z| \leqslant B$, $x \geqslant 2$.

This follows from Theorem 1 on using the identity

$$A_z(x) = \sum_{n \leqslant x} b_z(n) D_z\left(\frac{x}{n}\right).$$

THEOREM 3. *Under the assumptions of the preceding theorem, and if* $a_z(n) = \sum_{k=0}^{\infty} c_k(n) z^k$ *for* $|z| \leqslant B$, *we have for* $k \geqslant 1$,

$$C_k(x) = \sum_{n \leqslant x} c_k(n) = \frac{x}{(k-1)! \log x} \left[\frac{d^{k-1}}{dz^{k-1}} \frac{f(1,z)}{\Gamma(1+z)} (\log x)^z\right]_{z=0} + \\ + O\left(\frac{x (\log \log x)^k}{k! (\log x)^2}\right),$$

uniformly for $k \leqslant B \log \log x$, $x \geqslant 3$.

This follows from Theorem 2. On noting that

$$C_k(x) = \frac{1}{2\pi i} \int_C \frac{A_z(x)}{z^{k+1}} dz, \qquad (3)$$

where C is a circle around the origin with the radius $k/\log \log x$, one obtains the estimate.

THEOREM 4.* *Under the assumptions of the preceding theorem, and if the second derivative of* $\dfrac{f(1,z)}{\Gamma(1+z)}$ *is uniformly bounded for*

*We could here replace $(k-1)/\log \log x$ by $k/\log \log x$ which would correspond more closely to Sathe's results; however the remainder term will then become worse if $k = o(\log \log x)$ as the factor $k/(\log \log x)^2$ is replaced by $1/\log \log x$.

$|z| \leqslant B$, *we have, for* $k \geqslant 1$,

$$C_k(x) = \frac{x}{\log x} \cdot \frac{f\left(1, \frac{k-1}{\log\log x}\right)}{\Gamma\left(1 + \frac{k-1}{\log\log x}\right)} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} + $$

$$+ O\left(\frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \cdot \frac{k}{(\log\log x)^2}\right),$$

uniformly for $k \leqslant B \log\log x$, $x \geqslant 3$.

We may assume $k > 1$, since it is otherwise trivially containe[d in] the previous theorem. We then have only to evaluate the integr[al]

$$\frac{1}{2\pi i} \int_C \frac{f(1,z)}{\Gamma(1+z)} \frac{(\log x)^z}{z^k} dz, \qquad (4$$

extended over the circle $|z| = \frac{k-1}{\log\log x}$. Writing $g(z)$ for $\frac{f(1,z)}{\Gamma(1+z)}$

and $r = \frac{k-1}{\log\log x}$, we have

$$\frac{1}{2\pi i} \int_C g(z) \frac{(\log x)^z}{z^k} dz$$

$$= \frac{g(r)}{2\pi i} \int_C \frac{(\log x)^z}{z^k} dz + \frac{1}{2\pi i} \int_C \{g(z) - g(r) - g'(r)(z-r)\} \frac{(\log x)^z}{z^k} dz$$

$$= g(r) \frac{(\log\log x)^{k-1}}{(k-1)!} + O\left(\int_C |z-r|^2 \left|\frac{(\log x)^z}{z^k}\right| |dz|\right).$$

A simple estimation of the last integral gives the desired result f[or] the remainder term in Theorem 4.

The last two results contain all of Sathe's results by specializat[ion] as seen by taking $f(s, z)$ to be respectively

$$\prod_p \left(1 + \frac{z}{p^s}\right)\left(1 - \frac{1}{p^s}\right)^z, \prod_p \left(1 + \frac{z}{p^s - 1}\right)\left(1 - \frac{1}{p^s}\right)^z$$

and

$$\prod_p \left(1 - \frac{z}{p^s}\right)^{-1}\left(1 - \frac{1}{p^s}\right)^z.$$

$f(1, z)$ then respectively becomes $\pi_k(x)$, $\rho_k(x)$ and $\sigma_k(x)$ in Sathe's notation. It is easily seen that in the first two cases B may be taken arbitrarily large whereas in the third case B has to be taken less than 2, but can be taken arbitrarily close to 2. This accounts for the breakdown of the asymptotic formula for $\sigma_k(x)$ in case $k = [2 \log \log x]$ for instance.

It should be noted that the range of Theorem 3 can be extended somewhat if we only want the remainder term to be of less order than the main term, since we can then integrate over the circle $|z| = B$ in (3) and obtain this result for $k \leqslant B^* \log \log x$, where B^* is a larger constant than B. Applying this to the case of $\sigma_k(x)$ where $f(1, z)$ has a pole of first order at $z=2$, but otherwise is regular for $|z| < 3$, we can obtain an asymptotic formula which is valid for $k \leqslant B^* \log \log x$, where B^* is an arbitrary constant > 2. This however has a more complicated form since it contains an additional term, which is due to the pole at $z = 2$. This term is of smaller order than the others as long as $k \leqslant (2 - \epsilon) \log \log x$, but becomes later dominating, which explains the change in character of the asymptotic behaviour of $\sigma_k(x)$, so that for $(2 + \epsilon) \log \log x \leqslant k \leqslant B^* \log \log x$ actually $\sigma_k(x) \sim C\,(x \log x)/2^k$, where C is the residue of $g(z)$ at $z = 2$.

It should be noted that though Sathe's method is essentially elementary, the strong form of the prime-number theorem that he starts with cannot at present be obtained by elementary means.

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