# Moments, Concentration, and Entropy of Log-Concave Distributions 

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#### Abstract

We utilize and extend a simple and classical mechanism, combining log-concavity and majorization in the convex order to derive moments, concentration, and entropy inequalities for certain classes of log-concave distributions.


Keywords: Majorization, Log-concave, Concentration inequality, Entropy maximization.

## 1 Introduction

In this paper we explore how concavity of measure combined with the theory of majorization provides a simple mechanism for demonstrating several different types of integral inequalities. Using this technique we will derive concentration inequalities, moment comparison inequalities, and maximum entropy results. We will extend several classical and well known results for specific random variables, to variables with densities that are log-concave with respect to a given random variable.

Definition 1.1. A function $f: \mathbb{R} \rightarrow[0, \infty)$ is log-concave when

$$
f((1-t) x+t y) \geq f^{1-t}(x) f^{t}(y)
$$

holds for $x, y \in \mathbb{R}$ and $t \in[0,1]$.
Definition 1.2. A sequence $x: \mathbb{Z} \rightarrow[0, \infty)$ is log-concave when it possess contiguous support, that is $x_{m} x_{n}>0$ implies $x_{k}>0$ for $m \leq k \leq n$, and

$$
x_{k}^{2} \geq x_{k+1} x_{k-1}
$$

holds for all $k \in \mathbb{Z}$.
Note that a sequence $x: \mathbb{Z} \rightarrow[0, \infty)$ is log-concave if and only if there exists a log-concave function $f: \mathbb{R} \rightarrow[0, \infty)$ such that $f(k)=x_{k}$ for all $k \in \mathbb{Z}$.

Definition 1.3. For $Z$ an integer valued random variable with contiguous support, for a random variable $X$ we write $X \prec_{l c} Z$ and say that $X$ is log-concave with respect to $Z$ if the sequence

$$
y_{k}:= \begin{cases}\frac{\mathbb{P}(X=k)}{\mathbb{P}(Z=k)} & \text { if } \mathbb{P}(Z=k)>0 \\ 0 & \text { otherwise }\end{cases}
$$

is log-concave.

When $Z$ is a $\operatorname{binomial}(p, n)$, a random variable $X$ that is log-concave with respect to $Z$ is ultra log-concave of order $n$, written $\operatorname{ULC}(n)$, and equivalently satisfies,

$$
x_{k}^{2} \geq\left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right) x_{k-1} x_{k+1}
$$

for $1 \leq k \leq n-1$, where $x_{k}:=\mathbb{P}(X=k)$. When $Z$ is $\operatorname{Poisson}(\lambda)$, a random variable $X$ that is log-concave with respect to $Z$ is ultra log-concave (of order $\infty$ ), written ULC, and equivalently satisfies,

$$
x_{k}^{2} \geq\left(1+\frac{1}{k}\right) x_{k-1} x_{k+1}
$$

for $k \geq 1$. When $Z$ is $\operatorname{geometric}(p)$, a random variable $X$ that is log-concave with respect to $Z$ is simply log-concave (and supported on the non-negative integers), and equivalently satisfies,

$$
x_{k}^{2} \geq x_{k-1} x_{k+1},
$$

for $k \geq 1$.
Definition 1.4. For $Z$ an $\mathbb{R}$-valued random variable with convex support and density function $h$, for a random variable $X$ with density function $g$ we write $X \prec_{L C} Z$ if the function

$$
f(x):= \begin{cases}\frac{g(x)}{h(x)} & \text { if } h(x)>0 \\ 0 & \text { otherwise }\end{cases}
$$

is log-concave.
When $Z$ is Gaussian random variable with variance $1, X \log$-concave with respect to $Z$ are the so-called strongly log-concave random variables, and equivalently have densities that satisfy,

$$
g((1-t) x+t y) \geq e^{t(1-t)|x-y|^{2} / 2} g^{1-t}(x) g^{t}(y) .
$$

When $Z$ is exponentially distributed, $X \prec_{L C} Z$ is a (non-negative) log-concave random variable, with density satisfying

$$
g((1-t) x+t y) \geq g^{1-t}(x) g^{t}(y) .
$$

Definition 1.5. A random variable $X$ is majorized by $Z$ in the convex order, written $X \prec_{c x}$ Z, when

$$
\mathbb{E}[\varphi(X)] \leq \mathbb{E}[\varphi(Z)]
$$

holds for any convex function $\varphi$.
As we will only use majorization with respect to the convex order, by "majorization" we necessarily mean with respect to the convex order.

Our investigation begins with a simple observation, that for $X$ and $Z$ with matching expectations, $X \prec_{l c} Z$ or $X \prec_{L C} Z$ implies $X \prec_{c x} Z$. This result goes back at least to Whitt [16], and hinges on a classical inequality that can be found in Karlin and Studden [8] Lemma XI. 7.2., relating sign patterns of the difference densities to majorization (see also [18]).

Our first main result extends the tail bounds for sums of independent geometric and exponential random variables due to Janson [6], to sums of non-negative log-concave random variables.

Theorem 1.6. Let $X_{1}, \ldots, X_{n}, n \geq 1$, be independent discrete log-concave random variables on $\mathbb{N} \backslash\{0\}$. Denote $S_{n}=\sum_{i=1}^{n} X_{i}$. Then, for all $t \geq 1$,

$$
\mathbb{P}\left(S_{n} \geq t \mathbb{E}\left[S_{n}\right]\right) \leq e^{-\frac{\mathbb{E}\left[S_{n}\right]}{\max _{i} \mathbb{E}\left[X_{i}\right]}(t-1-\log (t))}
$$

and for all $t \leq 1$,

$$
\mathbb{P}\left(S_{n} \leq t \mathbb{E}\left[S_{n}\right]\right) \leq e^{-\frac{\mathbb{E}\left[S_{n}\right]}{\max _{i} \mathbb{E}\left[X_{i}\right]}(t-1-\log (t))}
$$

Taking $X_{i}$ to be geometric random variables recovers Theorem 2.1 and Theorem 3.1 of Janson [6].
Theorem 1.7. Let $X_{1}, \ldots, X_{n}, n \geq 1$, be independent positive continuous log-concave random variables, with $\mathbb{E}\left[X_{i}\right]=1$, and $\lambda_{i}>0$ with $\lambda:=\sum_{i=1}^{n} \lambda_{i}$ and $X_{\lambda}:=\sum_{i=1}^{n} \lambda_{i} X_{i}$. Then, all $t \geq 1$,

$$
\mathbb{P}\left(X_{\lambda} \geq t \lambda\right) \leq e^{-\lambda\left(\min _{i} \lambda_{i}\right)(t-1-\log (t))}
$$

and for all $t \leq 1$,

$$
\mathbb{P}\left(X_{\lambda} \leq t \lambda\right) \leq e^{-\lambda\left(\min _{i} \lambda_{i}\right)(t-1-\log (t))} .
$$

Note that taking $X_{i}$ to be i.i.d. exponential(1), and $\lambda_{i}=\frac{1}{a_{i}}$ recovers Theorem 5.1 (i) and (iii) of [6].

We extend the classical Chernoff-Hoeffding bounds [5] for independent Bernoulli sums, stated in terms of the relative entropy, to ultra log-concave random variables of order $n$.
Theorem 1.8. Let $X$ be a ULC(n) random variable. Let $\mu=\mathbb{E}[X] / n$. Then for $t \geq 0$,

$$
\begin{aligned}
& \mathbb{P}(X \geq(\mu+t) n) \leq e^{-n D(\mu+t \| \mu)} \\
& \mathbb{P}(X \leq(\mu-t) n) \leq e^{-n D(\mu-t| | \mu)}
\end{aligned}
$$

Here, $D(p \| q):=p \log \frac{p}{q}+(1-p) \log \frac{1-p}{1-q}$.
We also recover the recent concentration inequalities derived for ULC random variables in [1] used to generalize and improve the concentration inequalities for intrinsic volume random variables in [10] (see also [15]).
Theorem 1.9. Let $X$ be an ultra log-concave random variable. Then, for all $t \geq 0$,

$$
\mathbb{P}(X-\mathbb{E}[X] \geq t) \leq e^{-\frac{t^{2}}{2(t+\mathbb{E}[X])}},
$$

and

$$
\mathbb{P}(X-\mathbb{E}[X] \leq-t) \leq e^{-\frac{t^{2}}{2 \mathbb{E}[X]}}
$$

The notion of convex majorization will immediately give sharp comparisons between the expectation and other moments of log-concave random variables, simply by convexity (resp. concavity) of the map $x \mapsto x^{p}$ for $p \geq 1$ (resp. $p \in(0,1)$ ). Through a (to be proven) extension of Whitt [16] (see Theorem 2.7 below), we are able to derive new sharp moment comparison results for discrete log-concave random variables.

One of several results in this direction is the following theorem of Keilson [9], which we improve upon in the ultra log-concave case. These can be considered discrete analogs of the well known fact that in the continuous setting, the function $p \mapsto \frac{1}{\Gamma(p+1)} \int_{0}^{+\infty} t^{p} f(t) d t$ is log-concave whenever $f$ is log-concave. This also complements the recent confirmation in [13] of the conjectured log-concavity of the map $p \mapsto p \sum_{k} x_{k}^{p}$ for $x_{k}$ a monotone log-concave sequence (see [14]). We use the notation $(m)_{n}=\frac{m!}{(m-n)!}$, with the convention that $(m)_{n}=0$ if $m<n$.

Theorem 1.10 (Keilson [9]). Let $X$ be a discrete log-concave random variable supported on $\mathbb{N}$. Then the function

$$
\Phi: p \mapsto \frac{1}{p!} \mathbb{E}\left[(X)_{p}\right]
$$

is log-concave on $\mathbb{N}$, where $\mathbb{E}\left[(X)_{p}\right]$ are the factorial moments of $X$.
A strengthening is obtained under ultra log-concavity.
Theorem 1.11. For $X$ a $U L C(n)$ random variable,

$$
\Phi: p \mapsto \frac{\mathbb{E}\left[(X)_{p}\right]}{(n)_{p}}
$$

is log-concave, or equivalently

$$
\begin{equation*}
\mathbb{E}\left[(X)_{p}\right]^{2} \geq c_{n}(p) \mathbb{E}\left[(X)_{p+1}\right] \mathbb{E}\left[(X)_{p-1}\right] \tag{1}
\end{equation*}
$$

with $c_{n}(p)=\left(1+\frac{1}{n-p}\right)$. When $X$ is ULC, this is interpreted as $p \mapsto \mathbb{E}\left[(X)_{p}\right]$ is log-concave, or that (1) holds with $c_{\infty}(p)=1$ for all $p$.

As mentioned above the fact that log-concavity implies majorization was observed in [16] and was utilized by Yu in [18]. Yu used majorization techniques to simplify and slightly generalize results on the maximum entropy of compound Poisson distributions [7]. In particular Yu proved that for $X$ a ULC random variable and $Z$ a Poisson $(\mathbb{E}[X])$ random variable,

$$
H(X) \leq H(Z)
$$

where $H$ denotes the usual Shannon entropy. Before stating a generalization of this result, let us recall the definition of the Rényi entropy.
Definition 1.12. For a random variable $X$ with probability mass function $x_{k}:=\mathbb{P}(X=k)$, and $\alpha \in(0,1) \cup(1, \infty)$,

$$
H_{\alpha}(X):=\frac{1}{1-\alpha} \log \sum_{k} x_{k}^{\alpha} .
$$

Further, $H_{0}(X):=\log \left|\left\{x_{n}>0\right\}\right|$ with $|\cdot|$ denoting cardinality, $H_{1}(X)=H(X)$ is the usual Shannon entropy $-\sum_{k} x_{k} \log x_{k}$, and $H_{\infty}(X)=-\log \|x\|_{\infty}$, with $\|x\|_{\infty}:=\max _{k} x_{k}$. When $X$ is a continuous random variable with density function $f$ with respect to Lebesgue measure, we write

$$
h_{\alpha}(X):=\frac{1}{1-\alpha} \log \int f^{\alpha}(x) d x
$$

with $h_{0}(X):=\log |\{f>0\}|$ where $|\cdot|$ denotes volume, $h_{1}(X)=h(X)=-\int f(x) \log f(x) d x$, and $h_{\infty}(X)=-\log \|f\|_{\infty}$, with $\|f\|_{\infty}$ denoting the essential supremum of $f$ with respect to Lebesgue measure.
Theorem 1.13. Let $\alpha \leq 1$. For a discrete random variable $X$ such that $X \prec_{l c} Z$, if $Z$ is log-concave satisfying $\mathbb{E}[X]=\mathbb{E}[Z]$ then

$$
H_{\alpha}(X) \leq H_{\alpha}(Z)
$$

For a continuous random variable $X$ such that $X \prec_{L C} Z$, with $Z$ log-concave and satisfying $\mathbb{E}[X]=\mathbb{E}[Z]$,

$$
h_{\alpha}(X) \leq h_{\alpha}(Z)
$$

As a consequence, with $\alpha \leq 1$, the geometric distribution and exponential distribution have maximum $\alpha$-entropy among non-negative discrete and continuous log-concave distributions with fixed expectation, extending the classical fact that the geometric and exponential distributions have maximum Shannon entropy among all positive distributions of fixed expectation. Further the Poisson distribution has maximum $\alpha$-entropy for fixed expectation extending [18], and the $\operatorname{binomial}(p, n)$ has maximum $\alpha$-entropy among $\operatorname{ULC}(n)$ variables with expectation $p n$, extending the result in the Shannon case given by Yu [17] proven through "thinning techniques", which extended Harremoës [4] who had proven the same result for the subset of ULC $(n)$ consisting of independent Bernoulli sums of length $n$.

Let us outline the rest of the paper. In Section 2 we will deal with the technical aspects of the majorization results needed in both the continuous and discrete setting. The main technical result of this section is that if majorization is induced by relative log-concavity, then the majorization will be preserved under monotone maps. We then pursue with applications, in Section 3 concentration results are derived for the various classes of log-concave random variables. In Section 4 we will derive moment inequalities and in Section 5 we derive maximum entropy inequalities for the aforementioned log-concave classes.

## 2 Majorization of Log-Concave Distributions

The following lemma describes a simple sufficient condition for majorization. Intuitively, it says that majorization holds when the density function of $X$ and $Z$ "cross" twice.
Theorem 2.1. Suppose that $f$ and $g$ are non-negative probability density functions on $[0,+\infty)$ with respect to a Borel measure $\mu$ such that

$$
\int_{0}^{\infty} f(x) d \mu(x)=\int_{0}^{\infty} g(x) d \mu(x)=1
$$

and

$$
\int_{0}^{\infty} x f(x) d \mu(x)=\int_{0}^{\infty} x g(x) d \mu(x)<+\infty
$$

If there exists an interval $I \subseteq(0, \infty)$ such that $g(y) \leq f(y)$ for $y \in I$ while $g(y) \geq f(y)$ for $y \notin I$, then $\varphi:(0, \infty) \rightarrow \mathbb{R}$ convex implies,

$$
\int_{0}^{\infty} \varphi(x) f(x) d \mu(x) \leq \int_{0}^{\infty} \varphi(x) g(x) d \mu(x) .
$$

Proof. Let us assume that $X \sim f d \mu$ and define for $\lambda \in \mathbb{R}$,

$$
\Phi_{X}(\lambda):=\int[x-\lambda]_{+} f(x) d \mu(x)=\mathbb{E}\left[[X-\lambda]_{+}\right]
$$

where for $a \in \mathbb{R},(a)_{+}:=\max (a, 0)$. Then,

$$
\begin{aligned}
\Phi_{X}(\lambda) & =\int_{0}^{\infty} \mathbb{P}\left([X-\lambda]_{+}>t\right) d t \\
& =\int_{0}^{\infty} \mathbb{P}(X>\lambda+t) d t \\
& =\int_{\lambda}^{\infty} \mathbb{P}(X>t) d t .
\end{aligned}
$$

Hence, for a smooth compactly supported function $\phi$,

$$
\begin{aligned}
\int_{0}^{\infty} \phi^{\prime}(\lambda) \Phi_{X}(\lambda) d \lambda & =\int_{0}^{\infty}\left(\int_{0}^{t} \phi^{\prime}(\lambda) d \lambda\right) \mathbb{P}(X>t) d t \\
& =\int_{0}^{\infty}(\phi(t)-\phi(0)) \mathbb{P}(X>t) d t
\end{aligned}
$$

Thus, as the derivative $h^{\prime}$ of a distribution $h$ given by the relation $\left\langle h^{\prime}, \phi\right\rangle=-\langle h, \phi\rangle$,

$$
\frac{d}{d \lambda} \mathbb{E}\left[[X-\lambda]_{+}\right]=-\mathbb{P}(X>\lambda)+\delta_{0} \mathbb{E}[X]
$$

as distributions. Similarly,

$$
\begin{aligned}
\int_{0}^{\infty} \phi^{\prime}(\lambda) \mathbb{P}(X>\lambda) d \lambda & =\int_{0}^{\infty} \phi^{\prime}(\lambda)\left(\int_{0}^{\infty} \mathbb{1}_{(\lambda,+\infty)}(x) f(x) d \mu(x)\right) d \lambda \\
& =\int_{0}^{\infty}\left(\int_{0}^{x} \phi^{\prime}(\lambda) d \lambda\right) f(x) d \mu(x) \\
& =\int_{0}^{\infty}(\phi(x)-\phi(0)) f(x) d \mu(x) .
\end{aligned}
$$

That is, the distributional derivative of $\mathbb{P}(X>\lambda)$ is given by $\phi \mapsto-\int \phi f d \mu+\phi(0)$. Denoting

$$
\Psi(\lambda)=\int_{0}^{\infty}[x-\lambda]_{+}(g(x)-f(x)) d \mu(x),
$$

and repeating the above argument for a random variable $Z \sim g d \mu$ will show that, as distributional derivatives,

$$
\begin{aligned}
\Psi^{\prime}(\lambda) & =\mathbb{P}(X>\lambda)-\mathbb{P}(Z>\lambda) \\
\Psi^{\prime \prime}(\phi) & =\int \phi(x)(g-f)(x) d \mu(x)
\end{aligned}
$$

By assumption $\Psi(0)=\lim _{\lambda \rightarrow \infty} \Psi(\lambda)=0$ and further it follows from the computation above that $\Psi^{\prime}(0) \geq \lim _{\lambda \rightarrow \infty} \Psi^{\prime}(\lambda)=0$. Finally, $\Psi^{\prime \prime}$ is a non-negative measure on $I^{c}$, and a nonpositive measure on $I$. Thus $\Psi^{\prime}$ is non-increasing on $I$ and non-decreasing on $I^{c}$, and hence $\Psi^{\prime}$ is non-negative on $[0, a)$ and non-positive on $(a, \infty)$ for some $a>0$, and finally $\Psi \geq 0$.

If $\mu$ and $\nu$ are measures on a measurable space $(E, \mathcal{F})$, we consider the pushforward measures $T_{*} \mu$ and $T_{*} \nu$ as measures on the measurable space induced by $T,(T(E), T(\mathcal{F}))$, where the $\sigma$-algebra $T(\mathcal{F})$ is defined by $A \subseteq T(E)$ belongs to $T(\mathcal{F})$ if and only if $T^{-1}(A) \in \mathcal{F}$.
Proposition 2.2. Let $\mu$ and $\nu$ be Borel measures such that $\nu \ll \mu$. Then for any measurable function $T, T_{*} \nu \ll T_{*} \mu$. In particular, if $T_{*} \mu$ is $\sigma$-finite, then $\frac{d T_{*} \nu}{d T_{*} \mu}$ exists.
Proof. $T_{*} \mu(A)=\mu\left(T^{-1}(A)\right)=0$ implies $\nu\left(T^{-1}(A)\right)=T_{*} \nu(A)=0$, so that $T_{*} \nu \ll T_{*} \mu$ and the Radon-Nikodym theorem implies that $\frac{d T_{*} \nu}{d T_{*} \mu}$ exists if $T_{*} \mu$ is $\sigma$-finite.
Theorem 2.3. Let $\nu \ll \mu$ be $\sigma$-finite measures on $\mathbb{R}$ and $T$ be non-decreasing such that $T_{*} \mu$ is $\sigma$-finite, then on $T(\operatorname{supp}(\mu))$

$$
\frac{d T_{*} \nu}{d T_{*} \mu}(y):=f^{*}(y)= \begin{cases}\frac{\nu\left(T^{-1}\{y\}\right)}{\mu\left(T^{-1}\{y\}\right)} & \text { for } \#\left\{T^{-1}\{y\}\right\}>1 \\ \frac{d \nu}{d \mu} \circ T^{-1}(y) & \text { for } \#\left\{T^{-1}\{y\}\right\}=1\end{cases}
$$

where $x=T^{-1}(y)$ is the unique point such that $T(x)=y$ and we use the convention that $\frac{0}{0}:=0$.

Proof. Since $T$ is non-decreasing, we have that for $y \in T(\mathbb{R}), T^{-1}(\{y\})=\{x \in \mathbb{R}: T(x)=y\}$ is an interval (possibly reduced to a singleton). Therefore, there are only countably many $y$ 's such that $\# T^{-1}(\{y\})>1$. Let us enumerate such $y$ 's as $\left\{y_{i}\right\}_{i \in I}$, for some index set $I \subset \mathbb{N}$, and let us denote $K=\left\{y \in T(\mathbb{R}): \# T^{-1}(\{y\})=1\right\}$. Note that for all $i \in I$,

$$
\begin{aligned}
T_{*} \nu\left(\left\{y_{i}\right\}\right)=\nu\left(\left\{T^{-1}\left(\left\{y_{i}\right\}\right)\right\}\right) & =\frac{\nu\left(\left\{T^{-1}\left(\left\{y_{i}\right\}\right)\right\}\right)}{\mu\left(\left\{T^{-1}\left(\left\{y_{i}\right\}\right)\right\}\right)} \mu\left(\left\{T^{-1}\left(\left\{y_{i}\right\}\right)\right\}\right) \\
& =f^{*}\left(y_{i}\right) \mu\left(\left\{T^{-1}\left(\left\{y_{i}\right\}\right)\right\}\right) \\
& =\int_{\left\{y_{i}\right\}} f^{*} d T_{*} \mu .
\end{aligned}
$$

Therefore, for any Borel set $A \subset \mathbb{R}$,

$$
T_{*} \nu\left(A \cap K^{c}\right)=\sum_{i \in I: y_{i} \in A} T_{*} \nu\left(\left\{y_{i}\right\}\right)=\sum_{i \in I: y_{i} \in A} \int_{\left\{y_{i}\right\}} f^{*} d T_{*} \mu=\int_{A \cap K^{c}} f^{*} d T_{*} \mu
$$

On the other hand, since $T^{-1}$ defines a map on $K$, one may write for $y \in K, T^{-1}(\{y\})=$ $T^{-1}(y) \in \mathbb{R}$, so that if $x \in T^{-1}(K)$, we have $T^{-1}(T(x))=x$, and thus

$$
\frac{d \nu}{d \mu}(x)=\frac{d \nu}{d \mu}\left(T^{-1}(T(x))\right)=f^{*}(T(x))
$$

Therefore, for any Borel set $A \subset \mathbb{R}$,

$$
\begin{aligned}
T_{*} \nu(A \cap K)=\nu\left(T^{-1}(A \cap K)\right) & =\int 1_{A \cap K}(T(x)) \frac{d \nu}{d \mu}(x) d \mu(x) \\
& =\int 1_{A \cap K}(T(x)) f^{*}(T(x)) d \mu(x) \\
& =\int_{A \cap K} f^{*} d T_{*} \mu .
\end{aligned}
$$

We conclude by writing

$$
T_{*} \nu(A)=T_{*} \nu(A \cap K)+T_{*} \nu\left(A \cap K^{c}\right)=\int_{A \cap K} f^{*} d T_{*} \mu+\int_{A \cap K^{c}} f^{*} d T_{*} \mu=\int_{A} f^{*} d T_{*} \mu .
$$

Definition 2.4. For a set $S \subseteq \mathbb{R}$, a function $F: S \rightarrow \mathbb{R}$ has finitely many zero crossings if there exists a partition $\left\{S_{i}\right\}_{i=0}^{n}$ of $S$, with $S_{0}<\cdots<S_{n}$, such that for all $x \in S_{i}, y \in S_{i+1}$, $F(x) F(y)<0$ and for all $x, y \in S_{i}$,

$$
\begin{equation*}
F(x) F(y) \geq 0 . \tag{2}
\end{equation*}
$$

An $F$ with finitely many zero crossings has $N$-zero crossings, when there exists $\left\{S_{i}\right\}_{i=0}^{N} a$ minimal partition of $S$ satisfying (2).

Theorem 2.5. Suppose that $\nu$ and $\gamma$ have densities $f$ and $g$ on a set $S$ with respect to $\mu$, and that $f-g$ has $n$-zero crossings, and that $T$ is a non-decreasing function such that $T_{*} \mu$ is $\sigma$-finite, then $T_{*} \nu$ and $T_{*} \gamma$ have densities $f^{*}$ and $g^{*}$ on $T(S)$ with respect to $T_{*} \mu$ and $f^{*}-g^{*}$ has no more than $n$-zero crossings.

Proof. By Radon-Nikodym and Theorem 2.3

$$
f^{*}-g^{*}= \begin{cases}\frac{\nu\left(T^{-1}\{y\}\right)-\gamma\left(T^{-1}\{y\}\right)}{\mu\left(T^{-1}\left\{T^{\prime}\right)\right.} & \text { for } \#\left\{T^{-1}\{y\}\right\}>1  \tag{3}\\ (f-g) \circ T^{-1}(y) & \text { for } \#\left\{T^{-1}\{y\}\right\}=1 .\end{cases}
$$

Let $\left\{S_{i}\right\}_{i=0}^{n}$ denote a minimal partition of $S$ with respect to $F:=f-g$, so that $F\left(S_{i}\right) F\left(S_{i+1}\right) \leq$ 0 , with $s \in S_{i}$ and $s^{\prime} \in S_{i+1}$ such that $F(s) F\left(s^{\prime}\right)<0$. Define a partition of $T(S)$ by

$$
J_{0}=\left\{y \in T\left(S_{0}\right): \text { for all } s \in S_{0},\left(f^{*}-g^{*}\right)(y) F(s) \geq 0\right\}
$$

and for $i \geq 1$,

$$
J_{i}=\left\{y \in T\left(S_{i}\right): \text { for all } s \in S_{i},\left(f^{*}-g^{*}\right)(y) F\left(S_{i}\right) \geq 0\right\}-\left(\bigcup_{k=0}^{i-1} J_{k}\right)
$$

The $J_{i}$ are by definition disjoint. Further taking $y_{1}, y_{2} \in J_{i}$ and $s \in S_{i}$ such that $F(s) \neq 0$ we have by definition

$$
\left(f^{*}-g^{*}\right)\left(y_{1}\right)\left(f^{*}-g^{*}\right)\left(y_{2}\right)=\frac{\left(f^{*}-g^{*}\right)\left(y_{1}\right) F(s)\left(f^{*}-g^{*}\right)\left(y_{2}\right) F(s)}{F^{2}(s)} \geq 0 .
$$

Thus it suffices to show that $\bigcup_{i=0}^{n} J_{i}=T(S)$. Given $y \in T(S)$, then $y \in T\left(S_{i}\right)$ for some $i$ by definition, and by monotonicity of $T$ it follows that $y \in T\left(S_{k}\right) \cap \cdots \cap T\left(S_{k+l}\right)$ for some $k$ minimal and some $l \geq 0$ maximal. If $\left(f^{*}-g^{*}\right)(y) F(s) \geq 0$ for all $s \in S_{k}$, we are done, as by definition we then have $y \in J_{k}$. Thus assume that there exists $s \in S_{k}$ such that $\left(f^{*}-g^{*}\right)(y) F(s)<0$. Therefore, for all $t \in S_{k+1},\left(f^{*}-g^{*}\right)(y) F(t) \geq 0$ as $\left(f^{*}-g^{*}\right)(y) F(s) \leq 0$ and $F(t) F(s) \leq 0$. We need only prove that $y \in T\left(S_{k+1}\right)$ as it will follow that $y \in J_{k+1}$. However if $y \notin T\left(S_{k+1}\right)$ then $T^{-1}(\{y\}) \subseteq S_{k}$ so that (3) and the definition of the partition $\left\{S_{i}\right\}$ gives $\left(f^{*}-g^{*}\right)(y) F(s) \geq 0$ for any $s \in S_{k}$ which contradicts the assumption that there exists $s \in S_{k}$ such that $\left(f^{*}-g^{*}\right)(y) F(s)<0$. Thus $y \in J_{k+1}$ and the proof is complete.

For the next theorem, we say that a random variable $X$ is log-concave (resp. log-affine) with respect to a Borel measure $\mu$ if it admits a probability density function with respect to $\mu$ that is log-concave (resp. log-affine). In other words, if there exists a log-concave function $f$ such that

$$
d P_{X}=f d \mu
$$

Theorem 2.6. Let $\mu$ be a Borel measure on $[0,+\infty)$. For $X$ a non-negative random variable that is log-concave with respect to $\mu$, and $Z$ a non-negative random variable that is log-affine with respect to $\mu$ on the entire support of $\mu$, and satisfying $\mathbb{E}[Z]=\mathbb{E}[X]$, then

$$
X \prec_{c x} Z .
$$

Proof. We will show that the probability density function of $X$ and $Z$ with respect to $\mu$ have exactly 2 crossings and apply Theorem 2.1. Let us denote by $f$ the p.d.f. of $X$ and by $a$ the p.d.f. of $Z$. Since $f$ is $\log$-concave, and $a$ is $\log$-affine, $f$ and $a$ can have at most two crossings.

If $f=a$, there is nothing to prove, so let us assume that there exists $x$ such that $f(x) \neq$ $a(x)$. In this case, we claim $f$ and $a$ must have exactly two crossings. If there are no crossings then we have $f(x) \leq a(x)$ for all $x$ or $f(x) \geq a(x)$ for all $x$ with strict inequality for some $y$. In either case, this contradicts $\int f(x) d \mu(x)=\int a(x) d \mu(x)=1$.

To have exactly one crossing, would contradict $\mathbb{E}[X]=\mathbb{E}[Z]$. Indeed, say $a(x) \geq f(x)$ for $x \leq x_{1}$ and $a(x) \leq f(x)$ for $x>x_{1}$. But this would imply $\mathbb{P}(Z>t) \geq \mathbb{P}(X>t)$ for all $t>0$, with a strict inequality for some $t$ (else $X$ and $Z$ would be the same distribution) and hence

$$
\mathbb{E}[Z]=\int_{0}^{\infty} \mathbb{P}(Z>t) d t>\int_{0}^{\infty} \mathbb{P}(X>t) d t=\mathbb{E}[X]
$$

gives the contradiction. Thus applying Theorem 2.1, we obtain $X \prec_{c x} Z$.
The following generalization of Theorem 2.6 is needed in some applications, such as moments comparison (see Section 4).

Theorem 2.7. Let $\mu$ be a Borel measure on $[0,+\infty)$. Suppose that $X$ is log-concave with respect to $\mu$ and that $Z$ is log-affine with respect to $\mu$ on the entirety of the support of $\mu$. If $T$ is a non-decreasing function such that $T_{*} \mu$ is $\sigma$-finite and $\mathbb{E}[T(X)]=\mathbb{E}[T(Z)]$, then

$$
T(X) \prec_{c x} T(Z) .
$$

Proof. If $X$ is $\log$-concave and $Z$ is $\log$-affine with respect to a reference measure $\mu$, then their densities have at most two crossings. Since $T$ is non-decreasing and $T_{*} \mu$ is $\sigma$-finite, the densities of $T(X)$ and $T(Z)$ have at most two crossings by Theorem 2.5. We can then repeat the proof of Theorem 2.6. If $T(X)=T(Z)$ the proof is trivial, hence we may assume there is at least one crossing of $T(X)$ and $T(Z)$, and since exactly one crossing would again contradict $\mathbb{E}[T(X)]=\mathbb{E}[T(Z)]$ the proof is complete.

## 3 Concentration Inequalities

The first application of majorization is used toward deriving concentration inequalities. Let $X$ be a log-concave random variable with respect to a reference measure $\mu$. According to Theorem 2.6, if $Z$ is $\mu$-log-affine supported on the whole $\{\mu>0\}$ such that $\mathbb{E}[Z]=\mathbb{E}[X]$, then $X \prec_{c x} Z$. As the result, for all convex function $\varphi$,

$$
\begin{equation*}
\mathbb{E}[\varphi(X)] \leq \mathbb{E}[\varphi(Z)] . \tag{4}
\end{equation*}
$$

The following result demonstrates that Chernoff-type tail bounds on a random variable can be transferred to a random variable it majorizes. To this end we define for a random variable $X$,

$$
\Lambda_{X}(\lambda):=\log \mathbb{E}\left[e^{\lambda X}\right] .
$$

For a real valued function $f$ defined on an interval $I$, we denote by $f^{*}$ the Legendre transform,

$$
\begin{equation*}
f^{*}(t)=\sup _{\lambda \in I} \lambda t-f(\lambda), \tag{5}
\end{equation*}
$$

defined on $I^{*}$ the set of $t$ such that the supremum is finite.
Theorem 3.1. For $X \prec_{c x} Z$,

$$
\mathbb{P}(X \geq t) \leq \exp \left[-\Lambda_{+}^{*}(t)\right], \quad \mathbb{P}(X \leq t) \leq \exp \left[-\Lambda_{-}^{*}(t)\right],
$$

where $\Lambda_{+}$is the function $\Lambda_{Z}(\lambda)=\log \mathbb{E}\left[e^{\lambda Z}\right]$ restricted to $\lambda>0$ while $\Lambda_{-}$is the restriction of $\Lambda_{Z}$ to $\lambda<0$.

Proof. For $\lambda>0$, the standard approach through Markov's inequality gives

$$
\mathbb{P}(X \geq t)=\mathbb{P}\left(e^{\lambda X} \geq e^{\lambda t}\right) \leq e^{\Lambda_{X}(\lambda)-\lambda t}
$$

Since $\Lambda_{X}(t) \leq \Lambda_{Z}(t)$ follows from $X \prec_{c x} Z$, taking the infimum over $\lambda>0$ yields the result. Similarly, for $\lambda<0$,

$$
\mathbb{P}(X \leq t)=\mathbb{P}\left(e^{\lambda X} \geq e^{\lambda t}\right) \leq e^{\Lambda_{X}(\lambda)-\lambda t}
$$

Taking the infimum over $\lambda<0$ completes the proof.
For discrete log-concave distributions, that is, distributions that are log-concave with respect to the geometric distribution, we deduce the concentration inequalities in Theorem 1.6. First, we will have use for the following elementary lemma.

Lemma 3.2. For $Z$ a geometric $(p)$ or exponential( $p$ ) random variable, and $\theta<\lambda \leq p$,

$$
\mathbb{E}\left[e^{\theta Z}\right] \leq e^{-\frac{\lambda}{p} \log \left[1-\frac{\theta}{\lambda}\right]}
$$

We take the convention that $Z$ a geometric $(p)$ satisfies $\mathbb{P}(Z=k)=p(1-p)^{k-1}$ for $k \geq 1$, and the usual definition that $Z$ is exponential $(p)$ when it has density function $f(x)=p e^{-p x}$ for $x \geq 0$.

Proof. When $Z=Z_{E}$ is exponential $(p)$,

$$
\mathbb{E}\left[e^{\theta Z_{E}}\right]=\frac{p}{p-\theta}=e^{-\log \left[1-\frac{\theta}{p}\right]} \leq e^{-\frac{\lambda}{p} \log \left[1-\frac{\theta}{\lambda}\right]} .
$$

When $Z=Z_{G}$ is geometric $(p)$,

$$
\mathbb{E}\left[e^{\theta Z_{G}}\right]=\frac{p}{p-\left(1-e^{-\theta}\right)} \leq \frac{p}{p-\theta}=\mathbb{E}\left[e^{\theta Z_{E}}\right]
$$

which gives the result by the argument above.
Proof of Theorem 1.6. Let $Z_{i}, i=1, \ldots, n$, be independent geometric distributions with parameter $p_{i}=\frac{1}{\mathbb{E}\left[X_{i}\right]}$, chosen so that $\mathbb{E}\left[X_{i}\right]=\mathbb{E}\left[Z_{i}\right]$. Thus, by Theorem 2.6, $X_{i} \prec_{c x} Z_{i}$. Hence, the result follows from estimates of the moment generating function of $\sum_{i=1}^{n} Z_{i}$.

For all $0<\theta<\lambda:=\min _{i} p_{i}=\left(\max _{i} \mathbb{E}\left[X_{i}\right]\right)^{-1}$, applying the product structure of the moment generating function, $X_{i} \prec_{c x} Z_{i}$ applied to the convex function $x \mapsto e^{\theta x}$, and then Lemma 3.2,

$$
\mathbb{E}\left[e^{\theta S_{n}}\right] \leq \Pi_{i=1}^{n} \mathbb{E}\left[e^{\theta Z_{i}}\right] \leq e^{-\sum_{i=1}^{n} \frac{\lambda}{p_{i}} \log \left(1-\frac{\theta}{\lambda}\right)}=e^{-\lambda \mathbb{E}\left[S_{n}\right] \log \left(1-\frac{\theta}{\lambda}\right)} .
$$

Therefore, by Markov's inequality

$$
\mathbb{P}\left(S_{n} \geq t \mathbb{E}\left[S_{n}\right]\right) \leq e^{-\theta t \mathbb{E}\left[S_{n}\right]-\lambda \mathbb{E}\left[S_{n}\right] \log \left(1-\frac{\theta}{\lambda}\right)}
$$

Taking $\theta=\lambda\left(1-\frac{1}{t}\right)$ yields the result for the large deviation bound. A similar argument applied to negative $\theta$ yields the small deviation bound.

Taking $n=1$ in Theorem 1.6 gives the following corollary.

Corollary 3.3. Let $X$ be a discrete log-concave random variable on $\mathbb{N} \backslash\{0\}$. Then, for all $t \geq 1$,

$$
\mathbb{P}(X \geq t \mathbb{E}[X]) \leq t e^{1-t}
$$

and for all $t \leq 1$,

$$
\mathbb{P}(X \leq t \mathbb{E}[X]) \leq t e^{1-t}
$$

Remark 3.4. One may obtain concentration bounds for $\sum_{i=1}^{n} a_{i} X_{i}, a_{i}>0$, via the same majorization approach since if $X_{i} \prec_{c x} Z_{i}$, then for all $\theta \in \mathbb{R}$,

$$
\mathbb{E}\left[e^{\theta a_{i} X_{i}}\right] \leq \mathbb{E}\left[e^{\theta a_{i} Z_{i}}\right]
$$

For example, for log-concave distributions on $\mathbb{N} \backslash\{0\}$, following the proof of Theorem 1.6 yields

$$
\mathbb{P}\left(\sum_{i=1}^{n} a_{i} X_{i} \geq t \mathbb{E}\left[\sum_{i=1}^{n} a_{i} X_{i}\right]\right) \leq e^{-\left(\min _{i} a_{i} \mathbb{E}\left[X_{i}\right]\right) \mathbb{E}\left[\sum_{i=1}^{n} a_{i} X_{i}\right](t-1-\log (t))},
$$

and for all $t \leq 1$,

$$
\mathbb{P}\left(\sum_{i=1}^{n} a_{i} X_{i} \leq t \mathbb{E}\left[\sum_{i=1}^{n} a_{i} X_{i}\right]\right) \leq e^{-\left(\min _{i} a_{i} \mathbb{E}\left[X_{i}\right]\right) \mathbb{E}\left[\sum_{i=1}^{n} a_{i} X_{i}\right](t-1-\log (t))} .
$$

Remark 3.5. One may obtain similar concentration bounds for log-concave distributions on $\mathbb{N}$ by applying the majorization argument with the following variant of the geometric distribution, $\mathbb{P}(Z=k)=p(1-p)^{k}, k \geq 0$.

One may obtain the same bounds as in Remark 3.4 for continuous log-concave distribution on $[0,+\infty)$.

Proof of Theorem 1.7. According to Theorem 2.6, $Z_{i}$ exponential(1) majorizes a log-concave random variable $X_{i}$ of mean 1. Note that the moment generating function of $\sum_{i=1}^{n} a_{i} Z_{i}$ satisfies

$$
\mathbb{E}\left[e^{\theta \sum_{i=1}^{n} a_{i} X_{i}}\right] \leq \mathbb{E}\left[e^{\theta \sum_{i=1}^{n} a_{i} Z_{i}}\right]=e^{-\sum_{i=1}^{n} \log \left(1-\theta a_{i}\right)}
$$

and one may then reproduce the proof of Theorem 1.6.
Proof of Theorem 1.8. Let $X$ be ULC( $n$ ). Then by Theorem 2.6, $X \prec_{c x} Z$ where $Z$ is $\operatorname{Binomial}(n, p)$ with $p=\mathbb{E}[X] / n$. By Theorem 3.1,

$$
\mathbb{P}(X \geq(p+t) n) \leq \inf _{\lambda>0}\left((1-p) e^{-\lambda(p+t)}+p e^{-\lambda(p+t-1)}\right)^{n}
$$

Evaluating the right side at its minimizer, that is when $e^{\lambda}=\frac{(1-p)(p+t)}{(1-p-t) p}$, gives the result. The lower bounds are derived similarly.

Proof of Theorem 1.9. Let $Z$ be a Poisson distribution with parameter $\lambda=\mathbb{E}[X]$. Choosing the convex function $x \mapsto e^{t x}$ for arbitrary $t \in \mathbb{R}$ yields a pointwise dominance of moment generating functions

$$
\mathbb{E}\left[e^{t X}\right] \leq \mathbb{E}\left[e^{t Z}\right]=e^{\mathbb{E}[X]\left(e^{t}-1\right)} .
$$

The result then follows from a standard application of Markov's inequality as before.
Full details of the application of Markov's inequality alluded to above can be found in [1]. There the authors use an identification of extreme points satisfying a linear constraint (see [12]) to derive the same pointwise domination between the moment generating functions of $X$ and $Z$ attained through majorization here.

## 4 Moment Bounds

The next application of majorization is used toward providing comparisons between moments.
Corollary 4.1. Let $\mu$ be a Borel measure on $[0,+\infty)$. For a random variable $X$ log-concave with respect to $\mu$, and $\beta>\alpha$,

$$
\begin{equation*}
\mathbb{E}\left[X^{\beta}\right]^{\frac{1}{\beta}} \leq A_{\alpha, \beta} \mathbb{E}\left[X^{\alpha}\right]^{\frac{1}{\alpha}} \tag{6}
\end{equation*}
$$

where

$$
A_{\alpha, \beta}=\frac{\mathbb{E}\left[Z^{\beta}\right]^{\frac{1}{\beta}}}{\mathbb{E}\left[Z^{\alpha}\right]^{\frac{1}{\alpha}}}
$$

and $Z$ is the $\mu$-log-affine distribution with $\mathbb{E}\left[Z^{\alpha}\right]=\mathbb{E}\left[X^{\alpha}\right]$.
Proof. The result is obtained by taking $u(x)=x^{\alpha}$ and the convex function $\varphi(x)=x^{\frac{\beta}{\alpha}}$ in Theorem 2.7.

Let us understand the quantity $A_{\alpha, \beta}=\mathbb{E}\left[Z^{\beta}\right]^{\frac{1}{\beta}} / \mathbb{E}\left[Z^{\alpha}\right]^{\frac{1}{\alpha}}$ for discrete log-concave distributions (when the reference measure is geometric). There is no absolute comparison for all $Z$ $\log$-affine. For example, taking $\beta=2$ and $\alpha=1$, we have for $Z \sim(1-p) p^{k}, k \geq 0$,

$$
\begin{aligned}
\mathbb{E}[Z] & =\frac{p}{1-p} \\
\mathbb{E}\left[Z^{2}\right] & =\frac{p(1+p)}{(1-p)^{2}}
\end{aligned}
$$

Therefore,

$$
\frac{\mathbb{E}\left[Z^{2}\right]^{1 / 2}}{\mathbb{E}[Z]}=\frac{\sqrt{p(1+p)}}{1-p} \frac{1-p}{p}=\frac{\sqrt{1+p}}{\sqrt{p}} \longrightarrow_{p \rightarrow 0}+\infty
$$

Hence, the supremum over all $\log$-affine $Z$ is $+\infty$. Nonetheless, under a lower bound on $p$, one can have absolute comparison. Indeed, note that

$$
\mathbb{E}\left[Z^{\alpha}\right]=(1-p) \sum_{k \in \mathbb{N}} k^{\alpha} p^{k}=(1-p) \sum_{k \in \mathbb{N}} k^{\alpha} e^{-k \log (1 / p)}
$$

Also, we have

$$
\int_{0}^{+\infty} x^{\alpha} e^{-x \log (1 / p)} d x=\frac{\Gamma(\alpha+1)}{\log (1 / p)^{\alpha+1}}
$$

On the other hand,

$$
\begin{aligned}
\int_{0}^{+\infty} x^{\alpha} e^{-x \log (1 / p)} d x & =\sum_{k \in \mathbb{N}} \int_{k}^{k+1} x^{\alpha} e^{-x \log (1 / p)} d x \\
& \leq \sum_{k \in \mathbb{N}} e^{-k \log (1 / p)} \int_{k}^{k+1} x^{\alpha} d x \\
& =\frac{1}{\alpha+1}\left[\sum_{k \in \mathbb{N}}(k+1)^{\alpha+1} p^{k}-\sum_{k \in \mathbb{N}} k^{\alpha+1} p^{k}\right] \\
& =\frac{1}{\alpha+1} \frac{1}{p}\left[\mathbb{E}\left[Z^{\alpha+1}\right] .\right.
\end{aligned}
$$

Therefore,

$$
\mathbb{E}\left[Z^{\alpha+1}\right] \geq \Gamma(\alpha+2) \frac{p}{\log (1 / p)^{\alpha+1}}
$$

Similarly, we have

$$
\begin{aligned}
\int_{0}^{+\infty} x^{\alpha} e^{-x \log (1 / p)} d x & \geq \sum_{k \in \mathbb{N}} e^{-(k+1) \log (1 / p)} \int_{k}^{k+1} x^{\alpha} d x \\
& =\frac{1}{\alpha+1}\left[\sum_{k \in \mathbb{N}}(k+1)^{\alpha+1} p^{k+1}-\sum_{k \in \mathbb{N}} k^{\alpha+1} p^{k+1}\right] \\
& =\frac{1}{\alpha+1} \mathbb{E}\left[Z^{\alpha+1}\right] .
\end{aligned}
$$

Finally, we have the comparison

$$
p^{\frac{1}{\alpha+1}} \Gamma(\alpha+2)^{\frac{1}{\alpha+1}} \leq \log (1 / p) \mathbb{E}\left[Z^{\alpha+1}\right]^{\frac{1}{\alpha+1}} \leq \Gamma(\alpha+2)^{\frac{1}{\alpha+1}} .
$$

We deduce that for all $\alpha<\beta$,

$$
1 \leq A_{\alpha, \beta}=\frac{\mathbb{E}\left[Z^{\beta}\right]^{\frac{1}{\beta}}}{\mathbb{E}\left[Z^{\alpha}\right]^{\frac{1}{\alpha}}} \leq \frac{1}{p^{\frac{1}{\alpha}}} \frac{\Gamma(\beta+1)^{1 / \beta}}{\Gamma(\alpha+1)^{1 / \alpha}} .
$$

Now, let $X$ be a discrete log-concave random variable and assume that $\beta \geq \alpha \geq 1$. Then, the constraint $\mathbb{E}\left[X^{\alpha}\right]=\mathbb{E}\left[Z^{\alpha}\right]$ implies

$$
\mathbb{E}[X] \leq \mathbb{E}\left[X^{\alpha}\right]^{1 / \alpha}=\mathbb{E}\left[Z^{\alpha}\right]^{1 / \alpha} \leq \frac{\Gamma(\alpha+1)^{\frac{1}{\alpha}}}{\log (1 / p)}
$$

Therefore,

$$
\frac{1}{p} \leq e^{\frac{\Gamma(\alpha+1) \frac{1}{\alpha}}{\mathbb{E}[X]}} \leq e^{\frac{\alpha}{\mathbb{E}[X]}}
$$

From convex majorization, we also have that the constraint $\mathbb{E}\left[X^{\alpha}\right]=\mathbb{E}\left[Z^{\alpha}\right]$ implies

$$
\mathbb{E}\left[X^{\beta}\right] \leq \mathbb{E}\left[Z^{\beta}\right]
$$

We finally deduce that

$$
\mathbb{E}\left[X^{\beta}\right]^{\frac{1}{\beta}} \leq \mathbb{E}\left[Z^{\beta}\right]^{\frac{1}{\beta}}=\mathbb{E}\left[X^{\alpha}\right]^{\frac{1}{\alpha}} \frac{\mathbb{E}\left[Z^{\beta}\right]^{\frac{1}{\beta}}}{\mathbb{E}\left[Z^{\alpha}\right]^{\frac{1}{\alpha}}} \leq \frac{1}{p^{\frac{1}{\alpha}}} \frac{\Gamma(\beta+1)^{1 / \beta}}{\Gamma(\alpha+1)^{1 / \alpha}} \mathbb{E}\left[X^{\alpha}\right]^{\frac{1}{\alpha}} \leq \frac{\Gamma(\beta+1)^{1 / \beta}}{\Gamma(\alpha+1)^{1 / \alpha}} \mathbb{E}\left[X^{\alpha}\right]^{\frac{1}{\alpha}} e^{\frac{1}{\mathbb{E}[X]}}
$$

We thus have proved the following.
Corollary 4.2. Let $X$ be a discrete log-concave random variable on $\mathbb{N}$. Then, for all $1 \leq$ $\alpha \leq \beta$, we have

$$
\mathbb{E}\left[X^{\beta}\right]^{\frac{1}{\beta}} \leq \frac{\Gamma(\beta+1)^{1 / \beta}}{\Gamma(\alpha+1)^{1 / \alpha}} \mathbb{E}\left[X^{\alpha}\right]^{\frac{1}{\alpha}} e^{\frac{1}{\mathbb{E}[X]}} .
$$

In particular, if $\mathbb{E}[X] \geq 1$, we have

$$
\mathbb{E}\left[X^{\beta}\right]^{\frac{1}{\beta}} \leq c_{\alpha, \beta} \mathbb{E}\left[X^{\alpha}\right]^{\frac{1}{\alpha}},
$$

where $c_{\alpha, \beta}$ depends only on $\alpha$ and $\beta$. Moreover, one may take $c_{\alpha, \beta}=e \frac{\Gamma(\beta+1)^{1 / \beta}}{\Gamma(\alpha+1)^{1 / \alpha}}$.

We note that this recovers the well known inequality in the continuous setting.
Corollary 4.3. For $X$ a non-negative random variable with log-concave density $f$ with respect to the Lebesgue measure, then

$$
\mathbb{E}\left[X^{\beta}\right]^{\frac{1}{\beta}} \leq \frac{\Gamma(\beta+1)^{1 / \beta}}{\Gamma(\alpha+1)^{1 / \alpha}} \mathbb{E}\left[X^{\alpha}\right]^{\frac{1}{\alpha}} .
$$

Proof. If $\mathbb{E}[X]=0$, then $X=0$ and the proof is complete. Consider $\mathbb{E}[X]>0$. For $\varepsilon>0$ define the $\log$-concave probability sequence $f_{\varepsilon}: \mathbb{N} \rightarrow \mathbb{R}$ by $f_{\varepsilon}(n)=\frac{f(\varepsilon n)}{\sum_{m} f(\varepsilon m)}$ and let $X_{\varepsilon}$ be a random variable with such density. With the definition

$$
\psi_{p}(\varepsilon)=\frac{\mathbb{E}\left[X^{p}\right]^{\frac{1}{p}}}{\varepsilon \mathbb{E}\left[X_{\varepsilon}^{p}\right]^{\frac{1}{p}}},
$$

we have

$$
\begin{aligned}
\mathbb{E}\left[X^{\beta}\right]^{\frac{1}{\beta}}=\varepsilon \psi_{\beta}(\varepsilon) \mathbb{E}\left[X_{\varepsilon}^{\beta}\right]^{\frac{1}{\beta}} & \leq \varepsilon \psi_{\beta}(\varepsilon) \frac{\Gamma(\beta+1)^{1 / \beta}}{\Gamma(\alpha+1)^{1 / \alpha}} \mathbb{E}\left[X_{\varepsilon}^{\alpha}\right]^{\frac{1}{\alpha}} \exp \left[\frac{1}{\mathbb{E}\left[X_{\varepsilon}\right]}\right] \\
& =\frac{\psi_{\beta}(\varepsilon) \Gamma(\beta+1)^{1 / \beta}}{\psi_{\alpha}(\varepsilon) \Gamma(\alpha+1)^{1 / \alpha}} \mathbb{E}\left[X^{\alpha}\right]^{\frac{1}{\alpha}} \exp \left[\frac{\varepsilon \psi_{1}(\varepsilon)}{\mathbb{E}[X]}\right]
\end{aligned}
$$

Taking $\varepsilon \rightarrow 0$ completes the proof as $\psi_{p}(\varepsilon) \rightarrow 1$ for any $p>0$, since $\sum_{n}(n \varepsilon)^{p} f(n \varepsilon) \varepsilon$ is a Riemann sum approximation of $\int_{0}^{\infty} x^{p} f(x) d x$.

Remark 4.4. Corollary 4.3 can be derived via majorization as well by Corollary 4.1. It remains to note that

$$
A_{\alpha, \beta}=\frac{\mathbb{E}\left[Z^{\beta}\right]^{\frac{1}{\beta}}}{\mathbb{E}\left[Y^{\alpha}\right]^{\frac{1}{\alpha}}}=\frac{\Gamma(\beta+1)^{\frac{1}{\beta}}}{\Gamma(\alpha+1)^{\frac{1}{\alpha}}},
$$

when $Z$ is an exponential random variable.
Now, assume that $\alpha \leq 1$ and let $\beta \geq \alpha$. Note that for all integer valued random variable $X$ with p.m.f. $f$, we have

$$
\mathbb{E}\left[X^{\alpha}\right]=\sum_{k \geq 1} k^{\alpha} f(k) \leq \sum_{k \geq 1} k f(k)=\mathbb{E}[X] .
$$

Therefore, the constraint $\mathbb{E}\left[X^{\alpha}\right]=\mathbb{E}\left[Z^{\alpha}\right]$ implies

$$
\mathbb{E}\left[X^{\alpha}\right] \leq \mathbb{E}[Z]=\frac{p}{1-p}
$$

and thus

$$
\frac{1}{p} \leq 1+\frac{1}{\mathbb{E}\left[X^{\alpha}\right]}
$$

Since the constraint $\mathbb{E}\left[X^{\alpha}\right]=\mathbb{E}\left[Z^{\alpha}\right]$ also implies

$$
\mathbb{E}\left[X^{\beta}\right] \leq \mathbb{E}\left[Z^{\beta}\right]
$$

we have
$\mathbb{E}\left[X^{\beta}\right]^{\frac{1}{\beta}} \leq \frac{1}{p^{\frac{1}{\alpha}}} \frac{\Gamma(\beta+1)^{1 / \beta}}{\Gamma(\alpha+1)^{1 / \alpha}} \mathbb{E}\left[X^{\alpha}\right]^{\frac{1}{\alpha}} \leq \frac{\Gamma(\beta+1)^{1 / \beta}}{\Gamma(\alpha+1)^{1 / \alpha}}\left(\mathbb{E}\left[X^{\alpha}\right]+1\right)^{\frac{1}{\alpha}} \leq 2^{\frac{1}{\alpha}-1} \frac{\Gamma(\beta+1)^{1 / \beta}}{\Gamma(\alpha+1)^{1 / \alpha}}\left(\mathbb{E}\left[X^{\alpha}\right]^{\frac{1}{\alpha}}+1\right)$.
We thus have proved the following.

Corollary 4.5. Let $X$ be a discrete log-concave random variable on $\mathbb{N}$. Let $\alpha \in(0,1]$. Then, for all $\beta \geq \alpha$, we have

$$
\mathbb{E}\left[X^{\beta}\right]^{\frac{1}{\beta}} \leq 2^{\frac{1}{\alpha}-1} \frac{\Gamma(\beta+1)^{1 / \beta}}{\Gamma(\alpha+1)^{1 / \alpha}}\left(\mathbb{E}\left[X^{\alpha}\right]^{\frac{1}{\alpha}}+1\right) .
$$

For integer moments, one may have improvements. In particular, we recover a result of Keilson [9]. Recall the notation $(m)_{n}=\frac{m!}{(m-n)!}$ for $m \geq n$, with the convention that $(m)_{n}=0$ if $m<n$.

Proof of Theorem 1.10. Denote by $f$ the probability mass function of $X$. Denote, for $p \in \mathbb{N}$, $\Phi(p)=\frac{1}{p!} \mathbb{E}\left[(X)_{p}\right]$. We want to prove that for all $p \in \mathbb{N}$,

$$
\Phi(p+1) \geq \sqrt{\Phi(p) \Phi(p+2)}
$$

For this, fix $p \in \mathbb{N}$ and consider the measure $\mu$ defined as

$$
d \mu(k)=\frac{(k)_{p}}{\sum_{k \geq 0}(k)_{p} f(k)} d \mu^{\#}(k), \quad k \in \mathbb{N},
$$

where $d \mu^{\#}$ is the counting measure. Let $g$ be a log-affine sequence such that

$$
\begin{gathered}
\int g(k) d \mu(k)=\int f(k) d \mu(k)=1, \\
\int T(k) g(k) d \mu(k)=\int T(k) f(k) d \mu(k),
\end{gathered}
$$

where $T(k)=\frac{(k)_{p+1}}{(k)_{p}}=(k-(p+1)) 1_{\{k \geq p+2\}}$ is a non-decreasing function. Equivalently,

$$
\begin{aligned}
\mathbb{E}\left[(X)_{p}\right] & =\sum_{k \geq 0}(k)_{p} g(k), \\
\mathbb{E}\left[(X)_{p+1}\right] & =\sum_{k \geq 0}(k)_{p+1} g(k) .
\end{aligned}
$$

The above equations can also be written probabilistically. Writing $Y \sim f d \mu$ and $Z \sim g d \mu$, we have $\mathbb{E}[T(Y)]=\mathbb{E}[T(Z)]$. By Theorem 2.7 we deduce that $T(Y) \prec_{c x} T(Z)$, that is, for any convex function $\varphi$,

$$
\int \varphi(T(k)) f(k) d \mu(k) \leq \int \varphi(T(k)) g(k) d \mu(k) .
$$

Choosing $\varphi(k)=(k+(p+1)) \cdots k$ gives

$$
\mathbb{E}\left[(X)_{p+2}\right] \leq \sum_{k \geq 0}(k)_{p+2} g(k) .
$$

Writing $g(k)=c a^{k}$, we have for all $l \in \mathbb{N}$,

$$
\sum_{k \geq 0}(k)_{l} c a^{k}=c \sum_{k \geq 0} k \cdots(k-l+1) a^{k}=c l!\sum_{k \geq 0}\binom{k}{l} a^{k}=c l!\frac{a^{l}}{(1-a)^{l+1}} .
$$

Therefore, from the above identities, we have

$$
\Phi(p)=\frac{1}{p!} \sum_{k \geq 0}(k)_{p} c a^{k}=\frac{c}{1-a}\left(\frac{a}{1-a}\right)^{p},
$$

and

$$
\begin{equation*}
\Phi(p+1)=\frac{c}{1-a}\left(\frac{a}{1-a}\right)^{p+1} \tag{7}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\frac{\Phi(p+1)}{\Phi(p)}=\frac{a}{1-a} . \tag{8}
\end{equation*}
$$

Since

$$
\mathbb{E}\left[(X)_{p+2}\right] \leq \sum_{k \geq 0}(k)_{p+2} c a^{k}=c(p+2)!\frac{a^{p+2}}{(1-a)^{p+2+1}},
$$

we deduce by (7) and (8) that

$$
\Phi(p+2) \leq \frac{c}{1-a}\left(\frac{a}{1-a}\right)^{p+2}=\left(\frac{1-a}{a}\right)^{p+1} \Phi(p+1)\left(\frac{\Phi(p+1)}{\Phi(p)}\right)^{p+2}=\frac{\Phi(p+1)^{2}}{\Phi(p)}
$$

This gives the desired result.
For ultra log-concave random variables, we have the improvement given in Theorem 1.11.
Proof of Theorem 1.11. The proof of Theorem 1.10 can be repeated, one needs only to utilize that $Z \operatorname{binomial}(n, q)$ satisfies

$$
\mathbb{E}\left[(Z)_{r}\right]=(n)_{r} q^{r}
$$

for the $\operatorname{ULC}(n)$ case; while $Z \operatorname{Poisson}(\lambda)$ satisfies

$$
\mathbb{E}\left[(Z)_{r}\right]=\lambda^{r},
$$

for the ULC case.
One easily deduces from Theorem 1.10 the following moment inequalities

$$
\left(\frac{\mathbb{E}\left[(X)_{r+1}\right]}{(r+1)!}\right)^{\frac{1}{r+1}} \leq\left(\frac{\mathbb{E}\left[(X)_{r}\right]}{r!}\right)^{\frac{1}{r}}, \quad r \in \mathbb{N},
$$

holding for all discrete log-concave random variables. From Theorem 1.11 we deduce that for all $\operatorname{ULC}(n)$ random variables,

$$
\left(\frac{\mathbb{E}\left[(X)_{r+1}\right]}{(n)_{r+1}}\right)^{\frac{1}{r+1}} \leq\left(\frac{\mathbb{E}\left[(X)_{r}\right]}{(n)_{r}}\right)^{\frac{1}{r}}, \quad 0 \leq r \leq n-1
$$

In particular, ultra $\log$-concave random variables satisfy $\mathbb{E}\left[(X)_{r+1}\right]^{\frac{1}{r+1}} \leq \mathbb{E}\left[(X)_{r}\right]^{\frac{1}{r}}$, for $r \in \mathbb{N}$.

## 5 Maximum Entropy Distributions

It was shown in [7] that Shannon entropy is maximized by the Poisson distribution within the class of ultra log-concave distributions when matching expectation. The case of maximizing Shannon entropy within the class of log-concave distributions with respect to binomial on $\{0, \ldots, n\}$ was treated in [17], extending the result of [4] for Bernoulli sums.

An application of Theorem 2.6 together with the following lemma easily yield Rényi entropy maximization within subclasses of log-concave distributions.

Lemma 5.1. Let $X \sim f, Z \sim g$ be random variables where $f, g$ are densities with respect to the counting measure in the discrete case, or with respect to the Lebesgue measure in the continuous case. In order to prove $H_{\alpha}(X) \leq H_{\alpha}(Z)$ or $h_{\alpha}(X) \leq h_{\alpha}(Z)$, it suffices to prove

$$
\begin{aligned}
\mathbb{E}\left[g^{\alpha-1}(X)\right] & \leq \mathbb{E}\left[g^{\alpha-1}(Z)\right], \quad \text { if } \alpha \in(0,1), \\
-\mathbb{E}[\log (g(X))] & \leq-\mathbb{E}[\log (g(Z))], \quad \text { if } \alpha=1, \\
\mathbb{E}\left[g^{\alpha-1}(X)\right] & \geq \mathbb{E}\left[g^{\alpha-1}(Z)\right], \quad \text { if } \alpha \in(1, \infty) .
\end{aligned}
$$

The proof is an application of Hölder's inequality, and the non-negativity of the relative entropy. A proof can be found in detail as Lemma 3.25 of [11].

Proof of Theorem 1.13. In the discrete setting, denote by $g$ the probability mass function of $Z$. Consider $\widetilde{g}$ the piecewise linear extension of $g$ so that $\widetilde{g}$ is a log-concave function on $[0,+\infty)$ (see, e.g., [2] Proposition 5.1). Therefore the functions $\varphi(x)=\widetilde{g}^{\alpha-1}(x)$ when $\alpha \in(0,1)$ and $\psi(x)=-\log (\widetilde{g}(x))$ are convex. It remains to apply Theorem 2.6 together with Lemma 5.1. The proof in the continuous setting is similar and more straightforward.

Remark 5.2. Note that for Shannon entropy $(\alpha=1)$, the case of log-concave distributions on $\mathbb{N}$ (or on $[0, \infty)$ ) is not so interesting as the geometric (or exponential) distribution maximizes Shannon entropy when fixing expectation for all distributions supported on $\mathbb{N}$ (or on $[0, \infty$ ), see, e.g, [3].

When $\alpha>1$, the situation is more intricate. In fact, it turns out that Theorem 1.13 does not hold for $\alpha>1$. We state this fact in the next proposition.

Proposition 5.3. Let $\alpha>1$. Then, there exists a log-concave random variable $X$ such that

$$
H_{\alpha}(X)>H_{\alpha}(Z),
$$

where $Z$ is a geometric random variable satisfying $\mathbb{E}[Z]=\mathbb{E}[X]$.
Proof. Let $Z$ be a geometric distribution with parameter $p \in\left(\frac{1}{2}, 1\right)$ and let $X$ be a Bernoulli distribution with parameter $\frac{1-p}{p}$, so that $\mathbb{E}[X]=\mathbb{E}[Z]$. We have

$$
\begin{gathered}
e^{(1-\alpha) H_{\alpha}(X)}=\left(\frac{1-p}{p}\right)^{\alpha}+\left(\frac{2 p-1}{p}\right)^{\alpha} \\
e^{(1-\alpha) H_{\alpha}(Z)}=\frac{p^{\alpha}}{1-(1-p)^{\alpha}}
\end{gathered}
$$

We claim that as $p \rightarrow 1$,

$$
\left(\frac{1-p}{p}\right)^{\alpha}+\left(\frac{2 p-1}{p}\right)^{\alpha}<\frac{p^{\alpha}}{1-(1-p)^{\alpha}}
$$

and therefore $H_{\alpha}(X)>H_{\alpha}(Z)$. Indeed, the above inequality is equivalent to

$$
p^{2 \alpha}-\left[(1-p)^{\alpha}+(2 p-1)^{\alpha}\right]\left(1-(1-p)^{\alpha}\right)>0 .
$$

An application of L'hôpital's rule can be used to show that

$$
\lim _{p \rightarrow 1} \frac{p^{2 \alpha}-\left[(1-p)^{\alpha}+(2 p-1)^{\alpha}\right]\left(1-(1-p)^{\alpha}\right)}{(p-1)^{2}}=\alpha
$$

and the result follows.

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