Example 4.4.1. In this example we wish to reprove that $f(x) = x^2$ is continuous on $\mathfrak{R}$. Thus, we choose any real number a and an arbitrary $\varepsilon > 0$. We need to find $\delta > 0$ so that whenever $|x-a| < \delta$, we will have $|x^2 - a^2| < \varepsilon$. Although different approaches to the method of proof exist, we choose to argue by writing

$$\begin{aligned} |x^2 - a^2| &= |x-a||x+a| \\ &< |x-a|(1+2|a|) \qquad \text{if } |x-a| < 1, \\ &< \left(\frac{\varepsilon}{1+2|a|}\right)(1+2|a|) \qquad \text{if also } |x-a| < \frac{\varepsilon}{1+2|a|} \\ &= \varepsilon. \end{aligned}$$

Hence, pick $\delta = \min\left\{1, \dfrac{\varepsilon}{1+2|a|}\right\}$. Thus, whenever $|x-a| < \delta$, we have $|f(x) - f(a)| < \varepsilon$. Observe that for this particular function, δ is a real number that depends not only on ε but on $x = a$ as well. The larger the number a, the smaller must be the chosen δ. For example, suppose that $\varepsilon = \dfrac{1}{4}$. If $a = 3$, then the largest δ that can be chosen is $\delta = \dfrac{1}{28} \approx 0.0357$. But if $a = 300$, then the largest δ that can be chosen is $\delta = \dfrac{1}{2404} \approx 0.000416$. Why? Finally, notice that as one goes farther out on the x-axis, the x-coordinates that are close together produce functional values far apart. Consider two points close together, say, 1000 and 1000.01. They are within 0.01 unit of each other. However, $|f(1000) - f(1000.01)| \approx 20$. We shall soon see why this observation is important. □