Example 4.4.2. For contrast, consider the same function $f(x) = x^2$, but on a different domain. That is, consider the continuous function $f(x) = x^2$ on domain $D = (-2, 1]$. To prove formally, using Definition 4.1.2, that f is continuous at every point $x = a$ in its domain, pick an arbitrary $\varepsilon > 0$ and find $\delta > 0$ such that $|x^2 - a^2| < \varepsilon$ whenever $|x - a| < \delta$ with $x \in (-2, 1]$. To this end, we can write

$$\begin{aligned} \left|x^2 - a^2\right| &= |x - a||x + a| \\ &\leq |x - a|\big(|x| + |a|\big) \\ &\leq |x - a|(2 + |a|) \\ &< \frac{\varepsilon}{2 + |a|}\big(2 + |a|\big) = \varepsilon \end{aligned}$$

whenever $x \in (-2, 1]$ and $|x - a| < \dfrac{\varepsilon}{2 + |a|}$. Thus, pick $\delta = \dfrac{\varepsilon}{2 + |a|}$, guaranteeing continuity of f on this set. As in Example 4.4.1, note that this δ also depends on ε and the point a.

It is extremely important to observe that in the discussion above, someone else could have chosen to write

$$\begin{aligned} \left|x^2 - a^2\right| &= |x - a||x + a| \\ &\leq |x - a|\big(|x| + |a|\big) \\ &< |x - a|(4) < \frac{\varepsilon}{4}(4) = \varepsilon \end{aligned}$$

whenever $x \in (-2, 1]$ and $|x - a| < \dfrac{\varepsilon}{4}$. Their choice for δ would be $\dfrac{\varepsilon}{4}$, which is independent of the location of the point $x = a$. Now curiosity kicks in, could the same be done in