Example 4.4.1? Is there some clever way of writing a continuity proof for $f(x) = x^2$ with $x \in (-\infty, \infty)$, where δ would depend only on ε and not the point $x = a$? The answer is, no. The functions $f(x) = x^2$ with $x \in (-\infty, \infty)$ and $f(x) = x^2$ with $x \in (-2, 1]$ are quite different functions. The following definition will help us distinguish between these two types of functions. □

Definition 4.4.3. A function $f : D \to \Re$ is *uniformly continuous on a set* $E \subseteq D \subseteq \Re$ if and only if for any given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(t)| < \varepsilon$ for all $x, t \in E$ satisfying $|x - t| < \delta$. If f is uniformly continuous on its domain D, we simply say that f is *uniformly continuous*.

The continuous function in Example 4.4.1 is not a uniformly continuous function. However, the continuous function in Example 4.4.2 is uniformly continuous. Also, observe that it makes no sense to talk about uniform continuity at a point unless it is an isolated point. In looking at a uniformly continuous function, whenever two points are within δ the images are within ε of each other. This δ does not depend on the choice of x or t, whereas the δ in Definition 4.1.2 may. That is, if a function is continuous but not uniformly continuous, then the choice of δ will depend not only on ε but on x or t as well. In other words, a function that is continuous but not uniformly continuous on a domain D means that for any chosen point $a \in D$ and any arbitrary $\varepsilon > 0$, we can only find a $\delta > 0$ that depends on both a and ε. This seemingly minor difference in definitions creates a substantially different concept.