Example 4.4.5. Prove that $f(x) = x^2$ is not uniformly continuous.

Proof. Here the domain of f is assumed to be all of $\mathfrak{R}$. Restricting the domain may very well lead to a uniformly continuous function. See Example 4.4.2. Remark 4.4.4 will be used in proving that f is not uniformly continuous. Later, other methods will be considered.

We begin with a particular ε, say, $\varepsilon = 1$, and two sequences $\{x_n\}$ and $\{t_n\}$ with $x_n = n$ and $t_n = n + \frac{1}{n}$. Notice that the two sequences, although close together, involve large values. These particular sequences were chosen with the suspicion that the differences in their functional values would be far apart. So we have

$$|x_n - t_n| = \left| n - \left(n + \frac{1}{n} \right) \right| \leq \frac{1}{n} \quad \text{but} \quad \left| f(x_n) - f(t_n) \right| = \left| 2 + \frac{1}{n^2} \right| \geq 1.$$

Hence, by Remark 4.4.4, $f(x) = x^2$ with $x \in (-\infty, \infty)$ is not uniformly continuous. □