must be continuous as well. Why?

THEOREM 4.4.6. *If $D \subset \Re$ is a closed and bounded set, and a function $f : D \to \Re$ is continuous, then f is uniformly continuous.*

Proof. The fact that D is a closed and bounded domain is essential to this theorem. If this domain is altered in any way, then the result need not be true. But it still may be true. Why? We will prove this theorem by assuming that f is not uniformly continuous and then reach a contradiction. To this end, pick $\varepsilon > 0$ and two sequences $\{x_n\}$ and $\{t_n\}$ in D such that $|x_n - t_n| \leq \frac{1}{n}$, but $|f(x_n) - f(t_n)| \geq \varepsilon$. By the Bolzano–Weierstrass theorem for sequences, there exists a subsequence $\{x_{n_k}\}$ that converges to, say, $\alpha \in D$. Since f is continuous, we have that

$$\lim_{k\to\infty} f\left(x_{n_k}\right) = f(\alpha).$$

Furthermore,

$$\left|t_{n_k} - \alpha\right| \leq \left|t_{n_k} - x_{n_k}\right| + \left|x_{n_k} - \alpha\right| \leq \frac{1}{n_k} + \left|x_{n_k} - \alpha\right|.$$

Thus, we can conclude that the subsequence $\{t_{n_k}\}$ also converges to α. Again, due to continuity of f, we have that

$$\lim_{k\to\infty} f\left(t_{n_k}\right) = f(\alpha).$$

Hence,

$$\lim_{k\to\infty} \left|f\left(x_{n_k}\right) - f\left(t_{n_k}\right)\right| = 0,$$

contradicting the fact that $|f(x_n) - f(t_n)| \geq \varepsilon$. The proof is complete. □