Th: If $f : [c, +\infty) \longrightarrow \mathbb{R}$ is continuous, and $\lim\limits_{x \to +\infty} f(x)$ exists (and is finite), then f is uniformly cont. on $[c, +\infty)$

Pf: Assume $c > 1$ (the case $c \leq 1$ is left as an exercise)

Define $g(x) = \begin{cases} f\left(\frac{c}{1-x}\right), & x \in [0,1) \\ L, & x = 1 \end{cases}$,

where $L = \lim\limits_{x \to +\infty} f(x)$.

Since f is cont. on $[c, +\infty)$ and $\lim\limits_{x \to +\infty} f(x) = L$, it follows that

g is cont. on $[0,1]$.

From the previous th., g is therefore uniformly cont. on $[0,1]$ (since $[0,1]$ is closed and bounded)

Hence, by def.,

$\forall \varepsilon > 0, \exists \delta > 0, \forall u, v \in [0,1], |u-v| \leq \delta$
$\Rightarrow |f(u) - f(v)| \leq \varepsilon$