4.6* Projects

Part 1. Compact Sets

In Section 1.1 we defined what is meant by an infinite union of sets. Here we wish to generalize this idea and then use it to define compactness.

Let G be a given a nonempty (*indexing*) set, and for each element $\alpha \in G$ there is a corresponding set A_α. The set $\{A_\alpha \mid \alpha \in G\}$ is called an *(indexed) family of sets*. Now we can define $\bigcup_{\alpha\in G} A_\alpha$ to be the set S such that $x \in S$ if and only if $x \in A_\alpha$ for at least one $\alpha \in G$. Similarly, $\bigcap_{\alpha\in G} A_\alpha$ is the set S such that $x \in S$ if and only if $x \in A_\alpha$ for every $\alpha \in G$. Clearly, if $G = N$, then $\bigcup_{n\in N} A_n$ reduces to the union defined previously, which in Section 1.1 we denoted by $\bigcup_{n=1}^{\infty} A_n$.

Definition 4.6.1. An *open cover* of set E is an indexed family of open sets $\{A_\alpha\}$ in $\Re$ such that $E \subseteq \bigcup_\alpha A_\alpha$; that is, E is completely covered by all of sets A_α.

Consider the statement: The collection $\{A_\alpha\}$ of open sets is an open cover of a set E if and only if for each $a \in E$ there exists α^* such that $a \in A_{\alpha^*}$. Is this statement equivalent to Definition 4.6.1?

Example 4.6.2. A collection of intervals $A_1 = (0, 2)$, $A_2 = (1, 3)$, $A_3 = (2, 4), \ldots$ covers the interval $(0, \infty)$. □

Definition 4.6.3. A set E is *compact* if and only if every open cover of E has a finite subcover. That is, if $\{A_\alpha\}$ is any open cover of E, then there exist finitely many α's, say $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $E \subseteq \bigcup_{i=1}^{k} A_{\alpha_i}$.

Example 4.6.4.

(a) The interval $(0, 1]$ can be covered by the collection $A_3 = \left\{\left(-1, \frac{1}{2}\right), \left(-\frac{1}{2}, 2\right), (0, 3)\right\}$. Note that $(0, 1] \subset \left(-1, \frac{1}{2}\right) \cup \left(-\frac{1}{2}, 2\right) \cup (0, 3)$. Also this open cover of $(0, 1]$ has finite subcovers, namely A_3 itself or $\left\{\left(-1, \frac{1}{2}\right), (0, 3)\right\}$. But this fact does not make the interval $(0, 1]$ a compact set. The key word in Definition 4.6.3 is "every." Consider an open cover of $(0, 1]$ to be H, where $H = \left\{\left(\frac{1}{n}, 3\right) \mid n \in N\right\}$. The set H has no finite subset that will entirely cover the interval $(0, 1]$, thus H has no finite subcover. Hence, $(0, 1]$ is not compact.

(b) The open cover of $(0, \infty)$ as given in Example 4.6.2 has no finite subcover. Therefore, $(0, \infty)$ is not compact. □

Problem 4.6.5. Determine which of the given sets is compact.

(a) $(-\infty, 0]$

(b) $[-1, 2] \cup \{3\}$

(c) N