Th: (Heine-Borel)

A set $E \subset \mathbb{R}$ is compact

$\Longleftrightarrow$ E is closed and bounded

Pf: $\Longleftarrow$: Difficult, will not be treated.

$\Longrightarrow$: Assume E is compact.

. E bounded: Note $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-n, n)$.

Denote $I_n = (-n, n)$.

Since $E \subset \mathbb{R} = \bigcup_{n \in \mathbb{N}} I_n$, and $\{I_n\}$ is an open cover of E, and E is compact, $\exists k \in \mathbb{N}$, $\exists n_1, \dots, n_k \in \mathbb{N}$ s.t.

$$E \subset I_{n_1} \cup \dots \cup I_{n_k} = I_{\max(n_1, \dots, n_k)}$$

$\Longrightarrow \forall x \in E, \ |x| \le \max(n_1, \dots, n_k)$

$\Longrightarrow$ E is bounded.