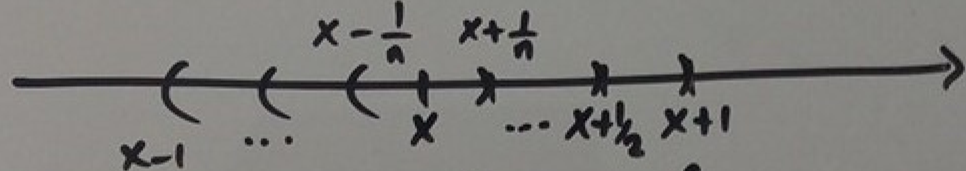
• **E closed:** By contradiction.

Assume E is not closed. Consider $\overline{E}$ the set of all accumulation point of E.

Since E is not closed, $E \neq \overline{E}$.

$\Longrightarrow \exists\, x \in \overline{E} \setminus E$.

Consider, for $n \geq 1$, $N_{1/n}(x) = \{ y \in \mathbb{R} : |y - x| < \frac{1}{n} \}$



Define, for $n \geq 1$, $A_n = \left(\overline{N_{1/n}(x)}\right)^c$ (the complement of the closure of $N_{1/n}(x)$)

Note that $A_n = (-\infty, x - \frac{1}{n}) \cup (x + \frac{1}{n}, +\infty)$.

Since $\overline{N_{1/n}(x)}$ is closed, A_n is open.

Note that $\bigcup_{n \geq 1} A_n = \mathbb{R} \setminus \{x\}$ (exercise).

Since $x \notin E$, then $E \subset \mathbb{R} \setminus \{x\} = \bigcup_{n \geq 1} A_n$.

$\Longrightarrow \{A_n\}$ is an open cover for S.