Thus, differentiability implies continuity, but as commented above, continuity does not imply differentiability. So, if we know that a function is differentiable, it must be continuous. Therefore, no continuity implies no differentiability, which is the contrapositive version of Theorem 5.1.7. Consider the next example for an illustration.

Example 5.1.8. Determine if the function $f(x) = \begin{cases} 2x+1 & \text{if } x \leq 0 \\ x^2+2x & \text{if } x > 0 \end{cases}$ is differentiable at $x = a$.

Answer. Perhaps we want to say that $f'(0) = 2$, but this is incorrect. The function f is not continuous at $x = 0$, so, using the contrapositive version of Theorem 5.1.7, we conclude that $f'(0)$ does not exist. □

A useful intuitive approach to a differentiable function is that it is continuous and its graph possesses neither a sharp point nor a vertical tangent line nor a singleton. Note that if f is differentiable at $x = a$, then

$$\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} = f'(a) \quad \text{and} \quad \lim_{2h\to 0}\frac{f(a+h)-f(a)}{h} = f'(a), \quad \text{but}$$

$$\lim_{h\to 0}\frac{f(a+2h)-f(a)}{h} = 2f'(a) \quad \text{and} \quad \lim_{h\to 0}\frac{f(a+h)-f(a)}{2h} = \frac{1}{2}f'(a).$$

See Exercises 9 and 10 and Example 5.1.12 for more on this topic.