this area. Even though we will assume that the reader is acquainted with the concepts and computations of derivatives, we will present the material formally with focus placed on proofs, discussions, and the inner relation of ideas given. New terminology may include continuous and/or uniform differentiability as well as topics in the "Project" section, where we study functions of bounded variation, absolutely continuous as well as convex functions.

5.1 Derivative of a Function

In the preceding introduction we referred to a tangent line[2] to a curve. Let us formally define what we mean by that.

Definition 5.1.1. Suppose that a function $f : D \to \Re$ with $D \subseteq \Re$, a is an accumulation point of D, and f is continuous at a. The *tangent line* to the graph(f) at $x = a$ is

(a) the line through the point $(a, f(a))$ having the slope $m(a)$, given by

$$m(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit is finite, or

(b) the line $x = a$, if $\lim_{x \to a^+} \dfrac{f(x) - f(a)}{x - a} = \pm\infty$ or $\lim_{x \to a^-} \dfrac{f(x) - f(a)}{x - a} = \pm\infty$.

If none of the limits in part (b) of Definition 5.1.1 are as listed, or a is not an accumulation point of D, then the graph (f) has no tangent line at the point $(a, f(a))$. The expression $\dfrac{f(x) - f(a)}{x - a}$, called a *difference quotient*, can be thought of as the slope of the *secant line* passing through the points $(a, f(a)$ and $(x, f(x))$. Secant lines will change as x changes. If x approaches a, the secant lines may or may not have a limit. If they do, the limit is the tangent line to the curve f at the point $x = a$. See Figure 5.1.1. If the tangent line is not vertical, then the value $m(a)$ from Definition 5.1.1 is called the derivative of f at $x = a$.