Proof. To verify this, we attempt to compute

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}.$$

Since the function f is defined differently to the right of zero than to the left, we must evaluate two limits. So we have

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1, \text{ and}$$
$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1.$$

Thus, $\lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist. Hence, $f'(0)$ does not exist. Observe in Figure 1.2.6, which illustrates f, that a sharp point exists at the origin. Thus, f has no tangent line at that point. □