THEOREM 5.4.8. *(Taylor's Theorem) Suppose that for any $n \in N$, a function f has $n+1$ derivatives in a neighborhood of $x = a$. If*

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

then $f(x) = p_n(x) + R_n(x)$, where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some c between x and a. If $x = a$, then $c = a$.

The function $p_n(x)$ is called the nth *Taylor polynomial of degree n centered about* $x = a$, and $f(x) = p_n(x) + R_n(x)$ is called *Taylor's formula with remainder*. The values $\frac{f^{(k)}(a)}{k!}$ are called the *Taylor coefficients*, and $R_n(x)$ is called the *remainder term*, which need not necessarily be small. $R_n(x)$, given in Theorem 5.4.8, is called *Lagrange's form of the remainder*. *Cauchy's form of the remainder* is given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a)$$

and is proven in Exercise 16 of Section 6.4. An integral form of $R_n(x)$ can be found in Exercise 14 of Section 6.4.

It should be noted that c is not necessarily unique. Also, note that c depends on the value of x. This means that $f^{(n+1)}(c)$ is not necessarily constant. Because of that, $R_n(x)$ need not be a polynomial of degree $n+1$. This is obvious from another perspective. Since $f(x) = p_n(x) + R_n(x)$, where $p_n(x)$ is a polynomial and $f(x)$ need not be, hence $R_n(x)$ also need not be a polynomial.