Proof of Taylor's theorem. Taylor's formula trivially is true if $x = a$. So we will assume that $x \neq a$ and prove that $R_n(x) = f(x) - p_n(x)$. To accomplish this, we use Rolle's theorem and prove that $R_n(b) = f(b) - p_n(b)$, for any $b \neq a$, where

$$R_n(b) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

for some c between a and b. So we apply Rolle's theorem to a new function F, defined by

$$F(x) = \left[f(x) + f'(x)(b-x) + \frac{f''(x)}{2!}(b-x)^2 + \cdots + \frac{f^{(n)}(x)}{n!}(b-x)^n \right] + \frac{R_n(b)}{(b-a)^{n+1}}(b-x)^{n+1}.$$

The reader should verify that $F(a) = f(b) = F(b)$ and that

$$F'(x) = \frac{f^{(n+1)}(x)}{n!}(b-x)^n - \frac{(n+1)R_n(b)}{(b-a)^{n+1}}(b-x)^n.$$