(See Exercise 16.) Therefore, by Rolle's theorem, there exists a c between a and b such that $F'(c) = 0$. Hence, we have

$$\frac{f^{(n+1)}(c)}{n!}(b-c)^n = \frac{(n+1)R_n(b)}{(b-a)^{n+1}}(b-c)^n,$$

which reduces to $R_n(b) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$. The proof is complete. □

It should be observed that in Taylor's theorem, $p_0(x), p_1(x), p_2(x), \ldots$ and so on, often become progressively closer and closer to the curve f near $x = a$ as n becomes larger and larger. However, after a certain point these polynomials may move drastically away from the function f. Note that p_1 is the tangent line to f at $x = a$, which generally will not be close to f for very long. In Figure 5.4.2, $f(x) = \sin x$ is graphed together with $p_n(x)$ for $1, 3, 5, \ldots, 15$ on the interval $[0, 7]$. What are the equations of these polynomials?

Taylor's theorem can be used to approximate certain values and/or prove inequalities. The next two examples demonstrate this. For further applications of Taylor's theorem, see Exercise 7 in Section 8.6.

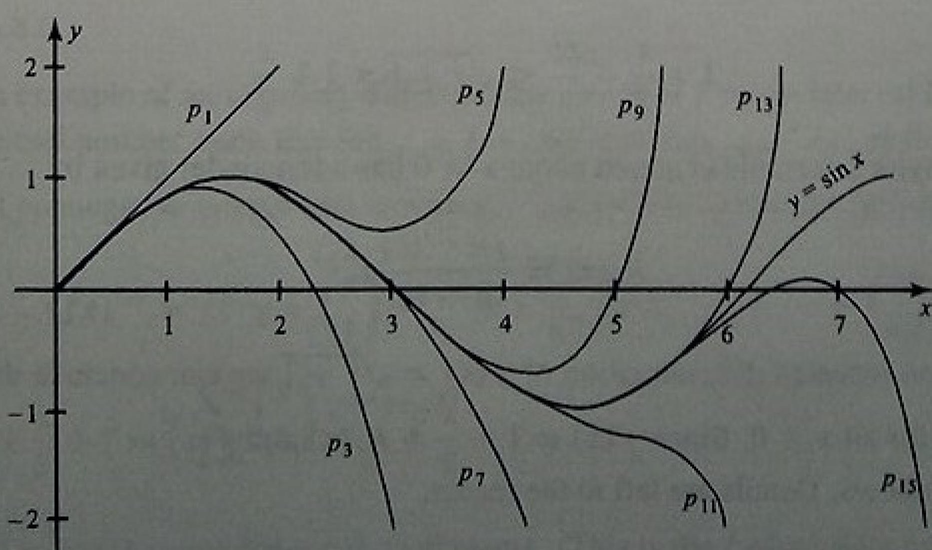

Figure 5.4.2