Example 5.4.10. For $x > 0$, use Taylor's theorem to prove that

$$1 + \frac{x}{2} - \frac{x^2}{8} \le \sqrt{x+1} \le 1 + \frac{x}{2}.$$

Proof. Since Taylor's formula centered about $x = 0$ has a remainder given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

with $c > 0$, upon repeated differentiation of $f(x) = \sqrt{x+1}$ we can conclude that $R_1(x) < 0$ and $R_2(x) > 0$ for all $x > 0$. Since $f(x) = 1 + \frac{x}{2} + R_1(x)$ and $f(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + R_2(x)$, the inequality follows. Details are left to the reader. □