Part 2. Lipschitz Condition

The Lipschitz condition, which is widely used in mathematics and was introduced briefly in Section 4.4, is the topic covered in this section.

Definition 5.7.6. We say that a function $f : D \to \Re$, with $D \subseteq \Re$ satisfies a *Lipschitz condition of order* α on D if and only if there exists a real number $L > 0$, called a *Lipschitz constant*, and a real constant $\alpha > 0$, such that

$$\left|f(x) - f(t)\right| \leq L|x - t|^{\alpha}$$

for all $x, t \in D$. If $\alpha = 1$, then we say that f satisfies the *Lipschitz condition*, or f is a *Lipschitz function*, or simply f is *Lipschitz*.

The next several theorems, remarks, and problems point out the relationship of the Lipschitz condition to other properties already discussed in this book.

THEOREM 5.7.7. *Suppose that a function f, defined on an interval I, satisfies the Lipschitz condition of order α. Then*

(a) *if $\alpha > 0$, f is uniformly continuous on I (see Theorem 4.4.11); and*

(b) *if $\alpha > 1$, f is differentiable on I. In fact, f must be constant on I.*

Remark 5.7.8. The converse of part (a) in Theorem 5.7.7 above does not hold. It can be shown that $f(x) = \sqrt[3]{x}$, on I = neighborhood of zero, is uniformly continuous but is not a Lipschitz function. (See Exercises 1(d) and 7(c) of Section 4.4.) Furthermore, a Lipschitz function need not be differentiable (see part (b) of Problem 5.7.11). Thus, part (b) of Theorem 5.7.7 fails if $\alpha = 1$. Also, worth noting is that if f is differentiable, it is also a Lipschitz function provided that the tangent lines are bounded away from the vertical line. Formally stated, this is the following theorem. □