

THEOREM 5.7.9. *Suppose that a function f is differentiable on an interval I . Then f' is bounded if and only if f is Lipschitz.*

Proof. (\Rightarrow) Suppose that $|f'(x)| \leq L$ for some real constant L with $x \in I$. Choose any $x_1, x_2 \in I$, with $x_1 < x_2$. By the mean value theorem, there exists a $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

Taking absolute values, we can rewrite this as

$$|f(x_1) - f(x_2)| = |f'(c)| |x_1 - x_2| \leq L|x_1 - x_2|.$$

Hence, f satisfies the Lipschitz condition.