Proof of Rolle's theorem. First, if f is a constant function, then the result follows from Exercise 2(a) of Section 5.1. Thus, assume that f is not constant. Since f is continuous on a closed and bounded interval $[a, b]$, it must attain a maximum and a minimum value (see Theorem 4.3.5). We will call them M and m, respectively. Note that $M \neq m$. The remainder of the proof is by contradiction. We will suppose that $f'(x) \neq 0$ for any $x \in (a, b)$, and look for a contradiction. Since f is differentiable on (a, b) and $f'(x) \neq 0$ at any $x \in (a, b)$, then f cannot have any extremum in (a, b). Why? Thus, f has an extremum at a and at b. But $f(a) = f(b)$. Hence $M = m$, which gives a contradiction. The proof is complete. □

THEOREM 5.3.3. *(Mean Value Theorem, sometimes known as Lagrange's Mean Value Theorem) Suppose that the following conditions are true for a function f:*

(a) *f is continuous on $[a, b]$ and,*

(b) *f is differentiable on (a, b).*

Then there exists some point $c \in (a, b)$ such that

$$f'(c) = \frac{f(a) - f(b)}{a - b} = \frac{f(b) - f(a)}{b - a}.$$