Example 5.3.5. Use the mean value theorem to prove:

(a) $\sin x \leq x$ for all $x \geq 0$. (See Exercise 51 from Section 1.9.)

(b) $ny^{n-1}(x - y) \leq x^n - y^n$ for $0 \leq y \leq x$ and n a natural number.

Proof of part (a). Let $f(x) = x - \sin x$. Then $f'(x) = 1 - \cos x$. So by the mean value theorem, there exists $c \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c).$$

Therefore, $f(x) - f(0) = xf'(c)$, which gives $x - \sin x = x(1 - \cos c) \geq 0$.

Proof of part (b). We define the function f by $f(x) = x^n$ with $n \geq 1$. Therefore, by the power rule, $f'(x) = nx^{n-1}$. Suppose that $0 \leq y < x$. By the mean value theorem, there exists $c \in (y, x)$ such that

$$\frac{x^n - y^n}{x - y} = nc^{n-1}.$$

Therefore, $x^n - y^n = nc^{n-1}(x - y) \geq ny^{n-1}(x - y)$. Since equality is true when $x = y$, we must have

$$ny^{n-1}(x - y) \leq x^n - y^n$$

for $0 \leq y \leq x$. Explain clearly where the conditions of x and y nonnegative are used. □