The following theorem is a generalization of the mean value result. Why? See Exercise 25.

THEOREM 5.3.8. *(Cauchy's Mean Value Theorem, sometimes known as the Generalized Mean Value Theorem) If functions f and g are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that*

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

Proof. Define a new function k by $k(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$ and then apply Rolle's theorem. Details of this proof are left to the reader in Exercise 22. □

Note that Cauchy's mean value theorem has a geometrical interpretation similar to that of the mean value theorem. However, in Cauchy's mean value theorem, we consider a "simple continuous curve" which is parametrized (see Section 9.4) by $x = g(t)$ and $y = f(t)$, with $t \in [a, b]$. The slope of the line segment joining the endpoints is given by

$$m = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Furthermore, the slope of the tangent line to the curve at any point $t = t_0$ is given by $\frac{f'(t_0)}{g'(t_0)}$. Cauchy's mean value theorem guarantees the existence of a value $c \in (a, b)$ for which the slope of the tangent line at c must equal m. Parametric equations are discussed in depth in Section 9.4.