If f is a differentiable function and f' is continuous, then we will say that f is *continuously differentiable* and write $f \in C^1$. If f is *n times differentiable* [i.e., $f^{(n)}(x)$ exists for all x in the domain of f], and if nth derivative is continuous, then we say that f is *n times continuously differentiable*. For example, the function f defined above is one-time differentiable but not one-time continuously differentiable. This function f, however, would be one-time continuously differentiable if we restricted it to an interval not containing zero. In fact, f would be *infinitely many times continuously differentiable*; that is, derivatives of all orders would exist.

Example 5.4.3. Consider the function f, defined by

$$f(x) = \begin{cases} \exp\left(-\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Show that f is an infinitely many times differentiable function.

Proof. If $x > 0$, then $f^{(n)}(x)$ exists for any $n \in N$, similarly for $x < 0$. Why? Therefore, we need only to verify that $f^{(n)}$ exists at $x = 0$ for every $n \in N$. If $n = 1$, we consider two limits:

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = 0$$

and

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{f(h)}{h} = \lim_{h \to 0^+} \frac{\exp(-\frac{1}{h})}{h} = \lim_{t \to \infty} \frac{t}{e^t} = 0.$$