Borell’s generalized Prékopa-Leindler inequality: A simple proof

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Abstract

We present a simple proof of Christer Borell’s general inequality in the Brunn-Minkowski theory. We then discuss applications of Borell’s inequality to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang.

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1 Introduction

Let us denote by $\text{supp}(f)$ the support of a function $f$. In [6] Christer Borell proved the following inequality (see [6, Theorem 2.1]), which we will call the Borell-Brunn-Minkowski inequality.

Theorem 1 (Borell-Brunn-Minkowski inequality). Let $f, g, h : \mathbb{R}^n \to [0, +\infty)$ be measurable functions. Let $\varphi = (\varphi_1, \ldots, \varphi_n) : \text{supp}(f) \times \text{supp}(g) \to \mathbb{R}^n$ be a continuously differentiable function with positive partial derivatives, such that $\varphi_k(x, y) = \varphi_k(x_k, y_k)$ for every $x = (x_1, \ldots, x_n) \in \text{supp}(f)$, $y = (y_1, \ldots, y_n) \in \text{supp}(g)$. Let $\Phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ be a continuous function, homogeneous of degree 1 and increasing in each variable. If the inequality

$$h(\varphi(x, y)) \prod_{k=1}^n \left( \frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \geq \Phi(f(x)) \prod_{k=1}^n \rho_k, g(y) \prod_{k=1}^n \eta_k$$

holds for every $x \in \text{supp}(f)$, for every $y \in \text{supp}(g)$, for every $\rho_1, \ldots, \rho_n > 0$ and for every $\eta_1, \ldots, \eta_n > 0$, then

$$\int h \geq \Phi \left( \int f, \int g \right).$$

C. Borell proved a slightly more general statement, involving an arbitrary number of functions. For simplicity, we restrict ourselves to the statement of Theorem 1.

Theorem 1 yields several important consequences. For example, applying Theorem 1 to indicators of compact sets (i.e. $f = 1_A$, $g = 1_B$, $h = 1_{\varphi(A,B)}$) yields the following generalized Brunn-Minkowski inequality.

Corollary 2 (Generalized Brunn-Minkowski inequality). Let $A, B$ be compact subsets of $\mathbb{R}^n$. Let $\varphi = (\varphi_1, \ldots, \varphi_n) : A \times B \to \mathbb{R}^n$ be a continuously differentiable function with positive partial derivatives, such that $\varphi_k(x, y) = \varphi_k(x_k, y_k)$ for every $x = (x_1, \ldots, x_n) \in A$, $y = (y_1, \ldots, y_n) \in B$. Let $\Phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ be a continuous function, homogeneous of degree 1 and increasing in each variable. If the inequality

$$\prod_{k=1}^n \left( \frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \geq \Phi(\prod_{k=1}^n \rho_k, \prod_{k=1}^n \eta_k)$$

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holds for every $\rho_1, \ldots, \rho_n, \eta_1, \ldots, \eta_n > 0$, then
\[
|\varphi(A, B)| \geq \Phi(|A|, |B|),
\]
where $| \cdot |$ denotes Lebesgue measure and $\varphi(A, B) = \{\varphi(x, y) : x \in A, y \in B\}$.

The classical Brunn-Minkowski inequality (see e.g. [23], [13]) follows from Corollary 2 by taking $\varphi(x, y) = x + y$, $x \in A, y \in B$, and $\Phi(a, b) = (a^{1/n} + b^{1/n})^n$, $a, b \geq 0$. Although the Brunn-Minkowski inequality goes back to more than a century ago, it still attracts a lot of attention (see e.g. [20], [11], [14], [18], [9], [10], [12], [15], [17]).

Theorem 4 also allows us to recover the so-called Borell-Brascamp-Lieb inequality. Let us denote by $M_s^\lambda(a, b)$ the $s$-mean of the real numbers $a, b \geq 0$ with weight $\lambda \in [0, 1]$, defined as
\[
M_s^\lambda(a, b) = ((1 - \lambda)a^s + \lambda b^s)^{1/s} \quad \text{if} \quad s \notin \{-\infty, 0, +\infty\},
\]
\[
M_{-\infty}^\lambda(a, b) = \min(a, b), \quad M_0^\lambda(a, b) = a^{1-\lambda} \lambda^{\lambda}, \quad M_{+\infty}^\lambda(a, b) = \max(a, b).
\]
We will need the following Hölder inequality (see e.g. [16]).

**Lemma 3 (Generalized Hölder inequality).** Let $\alpha, \beta, \gamma \in \mathbb{R} \cup \{+\infty\}$ such that $\beta + \gamma \geq 0$ and $1/\beta + 1/\gamma = 1/\alpha$. Then, for every $a, b, c, d \geq 0$ and $\lambda \in [0, 1]$,
\[
M_\alpha^\lambda(ac, bd) \leq M_\beta^\lambda(a, b) M_\gamma^\lambda(c, d).
\]

**Corollary 4 (Borell-Brascamp-Lieb inequality).** Let $\gamma \geq -1/n$, $\lambda \in [0, 1]$ and $f, g, h : \mathbb{R}^n \to [0, +\infty)$ be measurable functions. If the inequality
\[
h((1 - \lambda)x + \lambda y) \geq M_\gamma^\lambda(f(x), g(y))
\]
holds for every $x \in \text{supp}(f), y \in \text{supp}(g)$, then
\[
\int_{\mathbb{R}^n} h \geq M_{1+\gamma/n}^\lambda \left( \int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g \right).
\]

Corollary 4 follows from Theorem 1 by taking $\varphi(x, y) = (1 - \lambda)x + \lambda y, x \in \text{supp}(f), y \in \text{supp}(g)$, and $\Phi(a, b) = M_{1+\gamma/n}^\lambda(a, b), a, b \geq 0$. Indeed, using Lemma 3 one obtains that for every $x \in \text{supp}(f), y \in \text{supp}(g)$, and for every $\rho_1, \ldots, \rho_n, \eta_1, \ldots, \eta_n > 0,
\[
h(\varphi(x, y)) \Pi_{k=1}^n \left( \frac{\partial \varphi}{\partial x_k} \rho_k + \frac{\partial \varphi}{\partial y_k} \eta_k \right) = h((1 - \lambda)x + \lambda y) \Pi_{k=1}^n((1 - \lambda)\rho_k + \lambda \eta_k)
\]
\[
\geq M_\gamma^\lambda(f(x), g(y)) M_{1+\gamma/n}^\lambda(\Pi_{k=1}^n \rho_k, \Pi_{k=1}^n \eta_k)
\]
\[
\geq M_{1+\gamma/n}^\lambda(f(x) \Pi_{k=1}^n \rho_k, g(y) \Pi_{k=1}^n \eta_k)
\]
\[
= \Phi(f(x) \Pi_{k=1}^n \rho_k, g(y) \Pi_{k=1}^n \eta_k).
\]

Corollary 4 was independently proved by Borell (see [6] Theorem 3.1)], and by Brascamp and Lieb [3].

Another important consequence of the Borell-Brunn-Minkowski inequality is obtained when considering $\varphi$ to be nonlinear. Let us denote for $p = (p_1, \ldots, p_n) \in [-\infty, +\infty]^n$, $x = (x_1, \ldots, x_n) \in [0, +\infty]^n$ and $y = (y_1, \ldots, y_n) \in [0, +\infty]^n$,
\[
M_p^\lambda(x, y) = (M_{p_1}^\lambda(x_1, y_1), \ldots, M_{p_n}^\lambda(x_n, y_n)).
\]

**Corollary 5 (nonlinear extension of the Brunn-Minkowski inequality).** Let $p = (p_1, \ldots, p_n) \in [0, 1]^n$, $\gamma \geq -(\sum_{i=1}^n p_i^{-1})^{-1}$, $\lambda \in [0, 1]$, and $f, g, h : [0, +\infty)^n \to [0, +\infty)$ be measurable functions. If the inequality
\[
h(M_p^\lambda(x, y)) \geq M_\gamma^\lambda(f(x), g(y))
\]
holds for every $x \in \text{supp}(f), y \in \text{supp}(g)$, then
\[
\int_{[0, +\infty)^n} h \geq M_{1+\gamma/n}^\lambda \left( \int_{[0, +\infty)^n} f, \int_{[0, +\infty)^n} g \right).
\]
Corollary follows from Theorem by taking \(\varphi(x, y) = M^\lambda_p(x, y), x \in \text{supp}(f), y \in \text{supp}(g),\) and \(\Phi(a, b) = M^\lambda_{(\sum_{i=1}^n p_i^{-1} + \gamma^{-1})^{-1}}(a, b), a, b \geq 0.\) Indeed, using Lemma , one obtains that for every \(x \in \text{supp}(f), y \in \text{supp}(g),\) and for every \(\rho_1, \ldots, \rho_n, \eta_1, \ldots, \eta_n > 0,\)

\[
h(\varphi(x, y))\Pi^n_{k=1} \left(\frac{\partial \varphi}{\partial x_k} \rho_k + \frac{\partial \varphi}{\partial y_k} \eta_k\right) = h(M^\lambda_p(x, y))\Pi^n_{k=1} M^\lambda_{p_k}(x_k^{1-p_k}, y_k^{1-p_k})M_1(x_k^{p_k-1}\rho_k, y_k^{p_k-1}\eta_k)
\geq M^\lambda_\gamma(f(x), g(y))\Pi^n_{k=1} M^\lambda_{p_k}(\rho_k, \eta_k)
\geq M^\lambda_\gamma(f(x), g(y))M_1(\sum_{i=1}^n p_i^{-1})^{-1} (\Pi^n_{k=1} \rho_k, \Pi^n_{k=1} \eta_k)
\geq M^\lambda_{(\sum_{i=1}^n p_i^{-1} + \gamma^{-1})^{-1}} (f(x)\Pi^n_{k=1} \rho_k, g(y)\Pi^n_{k=1} \eta_k)
= \Phi(f(x)\Pi^n_{k=1} \rho_k, g(y)\Pi^n_{k=1} \eta_k).
\]

In the particular case where \(p = (0, \ldots, 0),\) Corollary was rediscovered by Ball . In the general case, Corollary was rediscovered by Uhrin .

Notice that the condition on \(p\) in Corollary is less restrictive in dimension 1. It reads as follows:

**Corollary 6** (nonlinear extension of the Brunn-Minkowski inequality on the line). Let \(p \leq 1, \gamma \geq -p,\) and \(\lambda \in [0, 1].\) Let \(f, g, h : [0, +\infty) \to [0, +\infty)\) be measurable functions such that for every \(x \in \text{supp}(f), y \in \text{supp}(g),\)

\[
h(M^\lambda_p(x, y)) \geq M^\lambda_\gamma(f(x), g(y)).
\]

Then,

\[
\int_0^{+\infty} h \geq M^\lambda_{\frac{1}{p} + \frac{1}{\gamma}^{-1}} \left(\int_0^{+\infty} f, \int_0^{+\infty} g\right).
\]

A simple proof of Corollary was recently given by Bobkov et al.

In section 2, we present a simple proof of Theorem based on mass transportation. In section 3, we discuss applications of the above inequalities to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang. We also prove an equivalence between the log-Brunn-Minkowski inequality and its possible extensions to convex measures (see section 3 for precise definitions).

## 2 A simple proof of the Borell-Brunn-Minkowski inequality

In this section, we present a simple proof of Theorem .

**Proof of Theorem**. The proof is done by induction on the dimension. To prove the theorem in dimension 1, we use a mass transportation argument.

**Step 1**: (In dimension 1)
First let us see that if \(\int f = 0\) or \(\int g = 0,\) then the result holds. Let us assume, without loss of generality, that \(\int g = 0.\) By taking \(\rho = 1,\) by letting \(\eta\) go to 0 and by using continuity and homogeneity of \(\Phi\) in the condition , one obtains

\[
h(\varphi(x, y))\frac{\partial \varphi}{\partial x} \geq \Phi(f(x), 0) = f(x)\Phi(1, 0).
\]

It follows that, for fixed \(y \in \text{supp}(g),\)

\[
\int h(z)dz \geq \int_{\varphi(\text{supp}(f), g)} h(z)dz = \int_{\text{supp}(f)} h(\varphi(x, y))\frac{\partial \varphi}{\partial x}dx \geq \int f\Phi(1, 0) = \Phi \left(\int f, \int g\right).
\]
A similar argument shows that the result holds if \( \int f = +\infty \) or \( \int g = +\infty \). Thus we assume thereafter that \( 0 < \int f < +\infty \) and \( 0 < \int g < +\infty \).

Let us show that one may assume that \( \int f = \int g = 1 \). Let us define, for \( x, y \in \mathbb{R} \) and \( a, b \geq 0 \),

\[
\tilde{f}(x) = f \left( \Phi \left( \int f, 0 \right) x \right) \Phi(1, 0), \quad \tilde{g}(x) = g \left( \Phi \left( 0, \int g \right) x \right) \Phi(0, 1),
\]

\[
\tilde{h}(x) = h \left( \Phi \left( \int f, \int g \right) x \right),
\]

\[
\tilde{\varphi}(x, y) = \Phi \left( \frac{\Phi(\int f, 0, x) \Phi(0, \int g) y}{\Phi(\int f, \int g)} \right), \quad \tilde{\Phi}(a, b) = \Phi \left( \frac{a \int f}{\Phi(\int f, \int g)} \frac{\int g}{\Phi(\int f, \int g)} \right).
\]

Let \( x \in \text{supp}(\tilde{f}) \), \( y \in \text{supp}(\tilde{g}) \), and let \( \tilde{\rho}, \tilde{\eta} > 0 \). One has,

\[
\tilde{h}(\tilde{\varphi}(x, y)) \left( \frac{\partial \tilde{\varphi}}{\partial x} \tilde{\rho} + \frac{\partial \tilde{\varphi}}{\partial y} \tilde{\eta} \right) \geq \Phi \left( f(\Phi(\int f, 0) x) \frac{\Phi(\int f, 0)}{\Phi(\int f, \int g)} \tilde{\rho}, g(\Phi(0, \int g) y) \frac{\Phi(0, \int g)}{\Phi(\int f, \int g)} \tilde{\eta} \right) = \Phi(f(x) \tilde{\rho}, g(y) \tilde{\eta}).
\]

Notice that the functions \( \tilde{\varphi} \) and \( \tilde{\Phi} \) satisfy the same assumptions as the functions \( \varphi \) and \( \Phi \) respectively, and that \( \int \tilde{f} = \int \tilde{g} = 1 \). If the result holds for functions of integral one, then

\[
\int \tilde{h}(w)dw \geq \tilde{\Phi}(1, 1) = 1.
\]

The change of variable \( w = z/\Phi(\int f, \int g) \) leads us to

\[
\int h(z)dz \geq \Phi \left( \int f, \int g \right).
\]

Assume now that \( \int f = \int g = 1 \). By standard approximation, one may assume that \( f \) and \( g \) are compactly supported positive Lipschitz functions (relying on the fact that \( \Phi \) is continuous and increasing in each coordinate, compare with [2] page 343). Thus there exists a non-decreasing map \( T : \text{supp}(f) \rightarrow \text{supp}(g) \) such that for every \( x \in \text{supp}(f) \),

\[
f(x) = g(T(x))T'(x),
\]

see e.g. [3], [25]. Since \( T \) is non-decreasing and \( \partial \varphi/\partial x, \partial \varphi/\partial y > 0 \), the function \( \Theta : \text{supp}(f) \rightarrow \varphi(\text{supp}(f), T(\text{supp}(f))) \) defined by \( \Theta(x) = \varphi(x, T(x)) \) is bijective. Hence the change of variable \( z = \Theta(x) \) is admissible and one has,

\[
\int h(z)dz \geq \int_{\text{supp}(f)} h(\varphi(x, T(x))) \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} T'(x) \right)dx \geq \int_{\text{supp}(f)} \Phi(f(x), g(T(x))T'(x))dx = \int \Phi(f(x), f(x))dx.
\]

Using homogeneity of \( \Phi \), one deduces that

\[
\int h \geq \Phi(1, 1) \int f(x)dx = \Phi \left( \int f, \int g \right).
\]

**Step 2 :** (Tensorization)

Let \( n \) be a positive integer and assume that Theorem 1 holds in \( \mathbb{R}^n \). Let \( f, g, h, \varphi, \Phi \) satisfying the assumptions of Theorem 1 in \( \mathbb{R}^{n+1} \). Recall that the inequality

\[
h(\varphi(x, y)) \prod_{k=1}^{n+1} \left( \frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \geq \Phi(f(x)) \prod_{k=1}^{n+1} \rho_k, g(y) \prod_{k=1}^{n+1} \eta_k,
\]

(2)
holds for every \( x \in \text{supp}(f), y \in \text{supp}(g) \), and for every \( \rho_1, \ldots, \rho_{n+1}, \eta_1, \ldots, \eta_{n+1} > 0 \). Let us define, for \( x_{n+1}, y_{n+1}, z_{n+1} \in \mathbb{R} \),

\[
F(x_{n+1}) = \int_{\mathbb{R}^n} f(x, x_{n+1})dx, \quad G(y_{n+1}) = \int_{\mathbb{R}^n} g(x, y_{n+1})dx, \quad H(z_{n+1}) = \int_{\mathbb{R}^n} h(x, z_{n+1})dx.
\]

Since \( \int f > 0, \int g > 0 \), the support of \( F \) and the support of \( G \) are nonempty. Let \( x_{n+1} \in \text{supp}(F), y_{n+1} \in \text{supp}(G) \), and let \( \rho_{n+1}, \eta_{n+1} > 0 \). Let us define, for \( x, y, z \in \mathbb{R}^n \),

\[
f_{x_{n+1}}(x) = f(x, x_{n+1})\rho_{n+1}, \quad g_{y_{n+1}}(y) = g(y, y_{n+1})\eta_{n+1}, \quad \varphi(x, y) = (\varphi_1(x_1, y_1), \ldots, \varphi_n(x_n, y_n)),
\]

\[
h_{\varphi_{n+1}}(z) = h(z, \varphi_{n+1}(x_{n+1}, y_{n+1})) \left( \frac{\partial \varphi_{n+1}}{\partial x_{n+1}} \rho_{n+1} + \frac{\partial \varphi_{n+1}}{\partial y_{n+1}} \eta_{n+1} \right).
\]

Let \( x \in \text{supp}(f_{x_{n+1}}), y \in \text{supp}(g_{y_{n+1}}) \), and let \( \rho_1, \ldots, \rho_n, \eta_1, \ldots, \eta_n > 0 \). One has

\[
h_{\varphi_{n+1}}(\varphi(x, y)) \prod_{k=1}^{n+1} \left( \frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) = h(\varphi(x, x_{n+1}, y, y_{n+1})) \prod_{k=1}^{n+1} \left( \frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) 
\geq \Phi(f(x, x_{n+1}) \prod_{k=1}^{n+1} \rho_k, g(y, y_{n+1}) \prod_{k=1}^{n+1} \eta_k) 
= \Phi(f_{x_{n+1}}(x) \prod_{k=1}^{n+1} \rho_k, g_{y_{n+1}}(y) \prod_{k=1}^{n+1} \eta_k),
\]

where the inequality follows from inequality \( 2 \). Hence, applying Theorem 1 in dimension \( n \), one has

\[
\int_{\mathbb{R}^n} h_{\varphi_{n+1}}(x)dx \geq \Phi \left( \int_{\mathbb{R}^n} f_{x_{n+1}}(x)dx, \int_{\mathbb{R}^n} g_{y_{n+1}}(x)dx \right).
\]

This yields that for every \( x_{n+1} \in \text{supp}(F), y_{n+1} \in \text{supp}(G) \), and for every \( \rho_{n+1}, \eta_{n+1} > 0 \),

\[
H(\varphi_{n+1}(x_{n+1}, y_{n+1})) \left( \frac{\partial \varphi_{n+1}}{\partial x_{n+1}} \rho_{n+1} + \frac{\partial \varphi_{n+1}}{\partial y_{n+1}} \eta_{n+1} \right) \geq \Phi(F(x_{n+1}), G(y_{n+1})).
\]

Hence, applying Theorem 1 in dimension 1, one has

\[
\int_{\mathbb{R}} H(x)dx \geq \Phi \left( \int_{\mathbb{R}} F(x)dx, \int_{\mathbb{R}} G(x)dx \right).
\]

This yields the desired inequality. \( \square \)

### 3 Applications to the log-Brunn-Minkowski inequality

In this section, we discuss applications of the above inequalities to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang \([7]\).

Recall that a convex body in \( \mathbb{R}^n \) is a compact convex subset of \( \mathbb{R}^n \) with nonempty interior. Böröczky et al. conjectured the following inequality.

**Conjecture 7** (log-Brunn-Minkowski inequality). Let \( K, L \) be symmetric convex bodies in \( \mathbb{R}^n \) and let \( \lambda \in [0, 1] \). Then,

\[
| (1 - \lambda) \cdot K \oplus_0 \lambda \cdot L | \geq | K |^{1-\lambda} | L |^\lambda.
\]

Here,

\[
(1 - \lambda) \cdot K \oplus_0 \lambda \cdot L = \{ x \in \mathbb{R}^n : \langle x, u \rangle \leq h_K(u)^{1-\lambda} h_L(u)^\lambda, \text{ for all } u \in S^{n-1} \},
\]

where \( S^{n-1} \) denotes the \( n \)-dimensional Euclidean unit sphere, \( h_K \) denotes the support function of \( K \), defined by \( h_K(u) = \max_{x \in K} \langle x, u \rangle \), and \( \cdot \) stands for Lebesgue measure.

Böröczky et al. \([7]\) proved that Conjecture 7 holds in the plane. Using Corollary 5 with \( p = (0, \ldots, 0) \), Saroglou \([21]\) proved that Conjecture 7 holds for unconditional convex bodies.
in $\mathbb{R}^n$ (a set $K \subset \mathbb{R}^n$ is unconditional if for every $(x_1, \ldots, x_n) \in K$ and for every $(\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$, one has $(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) \in K$).

Recall that a measure $\mu$ is s-concave, $s \in [-\infty, +\infty]$, if the inequality

$$\mu((1 - \lambda)A + \lambda B) \geq M_s^\lambda(\mu(A), \mu(B))$$

holds for all compact sets $A, B \subset \mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$ and for every $\lambda \in [0, 1]$ (see [5], [6]). The 0-concave measures are also called log-concave measures, and the $-\infty$-concave measures are also called convex measures. A function $f : \mathbb{R}^n \to [0, +\infty)$ is $\alpha$-concave, $\alpha \in [-\infty, +\infty]$, if the inequality

$$f((1 - \lambda)x + \lambda y) \geq M_\alpha^\lambda(f(x), f(y))$$

holds for every $x, y \in \mathbb{R}^n$ such that $f(x)f(y) > 0$ and for every $\lambda \in [0, 1]$.

Saroglou [22] recently proved that if the log-Brunn-Minkowski inequality holds, then the inequality

$$\mu((1 - \lambda) \cdot K \oplus_0 \lambda \cdot L) \geq \mu(K)^{1 - \lambda} \cdot \mu(L)^\lambda$$

holds for every symmetric log-concave measure $\mu$, for all symmetric convex bodies $K, L$ in $\mathbb{R}^n$ and for every $\lambda \in [0, 1]$.

An extension of the log-Brunn-Minkowski inequality for convex measures was proposed by the author in [19], and reads as follows:

**Conjecture 8.** Let $p \in [0, 1]$. Let $\mu$ be a symmetric measure in $\mathbb{R}^n$ that has an $\alpha$-concave density function, with $\alpha \geq -\frac{p}{n}$. Then for every symmetric convex body $K, L$ in $\mathbb{R}^n$ and for every $\lambda \in [0, 1]$,

$$\mu((1 - \lambda) \cdot K \oplus_p \lambda \cdot L) \geq M_{\alpha}^\lambda((\frac{p}{p + 1})^{\lambda} - 1)(\mu(K), \mu(L)). \quad (3)$$

Here,

$$(1 - \lambda) \cdot K \oplus_p \lambda \cdot L = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq M_{\alpha}^\lambda(h_K(u), h_L(u)), \text{ for all } u \in S^{n-1}\}.$$

In Conjecture 8 if $\alpha$ or $p$ is equal to 0, then $(n/p + 1/\alpha)^{-1}$ is defined by continuity and is equal to 0. Notice that Conjecture 7 is a particular case of Conjecture 8 when taking $\mu$ to be Lebesgue measure and $p = 0$.

By using Corollary 3 we will prove that Conjecture 7 implies Conjecture 8 when $\alpha \leq 1$, generalizing Saroglou’s result discussed earlier.

**Theorem 9.** If the log-Brunn-Minkowski inequality holds, then the inequality

$$\mu((1 - \lambda) \cdot K \oplus_p \lambda \cdot L) \geq M_{\alpha}^\lambda((\frac{p}{p + 1})^{\lambda} - 1)(\mu(K), \mu(L))$$

holds for every $p \in [0, 1]$, for every symmetric measure $\mu$ in $\mathbb{R}^n$ that has an $\alpha$-concave density function, with $1 \geq \alpha \geq -\frac{p}{n}$, for every symmetric convex body $K, L$ in $\mathbb{R}^n$ and for every $\lambda \in [0, 1]$.

**Proof.** Let $K_0, K_1$ be symmetric convex bodies in $\mathbb{R}^n$ and let $\lambda \in (0, 1)$. Let us denote $K_\lambda = (1 - \lambda) \cdot K_0 \oplus_p \lambda \cdot K_1$ and let us denote by $\psi$ the density function of $\mu$. Let us define, for $t > 0$, $h(t) = |K_\lambda \cap \{\psi \geq t\}|$, $f(t) = |K_0 \cap \{\psi \geq t\}|$ and $g(t) = |K_1 \cap \{\psi \geq t\}|$. Notice that

$$\mu(K_\lambda) = \int_{K_\lambda} \psi(x)dx = \int_{K_\lambda} \int_0^{\psi(x)} dt dx = \int_0^{+\infty} |K_\lambda \cap \{\psi \geq t\}| = \int_0^{+\infty} h(t)dt.$$

Similarly, one has

$$\mu(K_0) = \int_0^{+\infty} f(t)dt,$$  \hfill  \mu(K_1) = \int_0^{+\infty} g(t)dt.$$
Let \( t, s > 0 \) such that the sets \( \{ \psi \geq t \} \) and \( \{ \psi \geq s \} \) are nonempty. Let us denote \( L_0 = \{ \psi \geq t \} \), \( L_1 = \{ \psi \geq s \} \) and \( L_\lambda = \{ \psi \geq M_\lambda(t, s) \} \). If \( x \in L_0 \) and \( y \in L_1 \), then \( \psi((1 - \lambda)x + \lambda y) \geq M_\lambda(\psi(x), \psi(y)) \geq M_\lambda(t, s) \). Hence,

\[
L_\lambda \supset (1 - \lambda)L_0 + \lambda L_1 \supset (1 - \lambda) \cdot L_0 \oplus_p \lambda \cdot L_1,
\]

the last inclusion following from the fact that \( p \leq 1 \). We deduce that

\[
K_\lambda \cap L_\lambda \supset ((1 - \lambda) \cdot K_0 \oplus_p \lambda \cdot K_1) \cap ((1 - \lambda) \cdot L_0 \oplus_p \lambda \cdot L_1) \supset (1 - \lambda) \cdot (K_0 \cap L_0) \oplus_p \lambda \cdot (K_1 \cap L_1).
\]

Hence,

\[
h(M_\lambda(t, s)) = |K_\lambda \cap L_\lambda| \geq |(1 - \lambda) \cdot (K_0 \cap L_0) \oplus_p \lambda \cdot (K_1 \cap L_1)| \geq M^{\lambda}(f(t), g(s)),
\]

the last inequality is valid for \( p \geq 0 \) and follows from the log-Brunn-Minkowski inequality by using homogeneity of Lebesgue measure (see [7, beginning of section 3]). Thus we may apply Corollary [6] to conclude that

\[
\mu(K_\lambda) = \int_0^{+\infty} h \geq M^{\lambda}_{(\frac{\alpha}{\beta} + \frac{1}{\beta})^{-1}} \left( \int_0^{+\infty} f, \int_0^{+\infty} g \right) = M^{\lambda}_{(\frac{\alpha}{\beta} + \frac{1}{\beta})^{-1}}(\mu(K_0), \mu(K_1)).
\]

Since the log-Brunn-Minkowski inequality holds true in the plane, we deduce that Conjecture [8] holds true in the plane (with the restriction \( \alpha \leq 1 \)). Notice that Conjecture [8] holds true in the unconditional case as a consequence of Corollary [5] (see [19]).

References


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