Borell's generalized Prékopa-Leindler inequality: A simple proof

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Abstract

We present a simple proof of Christer Borell's general inequality in the Brunn-Minkowski theory. We then discuss applications of Borell's inequality to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang.

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1 Introduction

Let us denote by $\mathrm{supp}(f)$ the support of a function f. In [6] Christer Borell proved the following inequality (see [6, Theorem 2.1]), which we will call the Borell-Brunn-Minkowski inequality.

Theorem 1 (Borell-Brunn-Minkowski inequality). *Let $f, g, h : \mathbb{R}^n \to [0, +\infty)$ be measurable functions. Let $\varphi = (\varphi_1, \dots, \varphi_n) : \mathrm{supp}(f) \times \mathrm{supp}(g) \to \mathbb{R}^n$ be a continuously differentiable function with positive partial derivatives, such that $\varphi_k(x, y) = \varphi_k(x_k, y_k)$ for every $x = (x_1, \dots, x_n) \in \mathrm{supp}(f)$, $y = (y_1, \dots, y_n) \in \mathrm{supp}(g)$. Let $\Phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ be a continuous function, homogeneous of degree 1 and increasing in each variable. If the inequality*

$$h(\varphi(x,y))\Pi_{k=1}^n \left(\frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \geq \Phi(f(x)\Pi_{k=1}^n \rho_k, g(y)\Pi_{k=1}^n \eta_k) \qquad (1)$$

holds for every $x \in \mathrm{supp}(f)$, for every $y \in \mathrm{supp}(g)$, for every $\rho_1, \dots, \rho_n > 0$ and for every $\eta_1, \dots, \eta_n > 0$, then

$$\int h \geq \Phi\left(\int f, \int g \right).$$

C. Borell proved a slightly more general statement, involving an arbitrary number of functions. For simplicity, we restrict ourselves to the statement of Theorem 1.

Theorem 1 yields several important consequences. For example, applying Theorem 1 to indicators of compact sets (i.e. $f = 1_A$, $g = 1_B$, $h = 1_{\varphi(A,B)}$) yields the following generalized Brunn-Minkowski inequality.

Corollary 2 (Generalized Brunn-Minkowski inequality). *Let A, B be compact subsets of $\mathbb{R}^n$. Let $\varphi = (\varphi_1, \dots, \varphi_n) : A \times B \to \mathbb{R}^n$ be a continuously differentiable function with positive partial derivatives, such that $\varphi_k(x, y) = \varphi_k(x_k, y_k)$ for every $x = (x_1, \dots, x_n) \in A$, $y = (y_1, \dots, y_n) \in B$. Let $\Phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ be a continuous function, homogeneous of degree 1 and increasing in each variable. If the inequality*

$$\Pi_{k=1}^n \left(\frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \geq \Phi(\Pi_{k=1}^n \rho_k, \Pi_{k=1}^n \eta_k)$$

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holds for every $\rho_1, \ldots, \rho_n, \eta_1, \ldots, \eta_n > 0$*, then*

$$|\varphi(A, B)| \geq \Phi(|A|, |B|),$$

where $|\cdot|$ *denotes Lebesgue measure and* $\varphi(A, B) = \{\varphi(x, y) : x \in A, y \in B\}$.

The classical Brunn-Minkowski inequality (see e.g. [23], [13]) follows from Corollary 2 by taking $\varphi(x, y) = x + y$, $x \in A, y \in B$, and $\Phi(a, b) = (a^{1/n} + b^{1/n})^n$, $a, b \geq 0$. Although the Brunn-Minkowski inequality goes back to more than a century ago, it still attracts a lot of attention (see e.g. [20], [11], [14], [18], [9], [10], [12], [15], [17]).

Theorem 1 also allows us to recover the so-called Borell-Brascamp-Lieb inequality. Let us denote by $M_s^\lambda(a, b)$ the s-mean of the real numbers $a, b \geq 0$ with weight $\lambda \in [0, 1]$, defined as

$$M_s^\lambda(a, b) = ((1-\lambda)a^s + \lambda b^s)^{\frac{1}{s}} \quad \text{if } s \notin \{-\infty, 0, +\infty\},$$

$M_{-\infty}^\lambda(a, b) = \min(a, b)$, $M_0^\lambda(a, b) = a^{1-\lambda} b^\lambda$, $M_{+\infty}^\lambda(a, b) = \max(a, b)$. We will need the following Hölder inequality (see e.g. [16]).

Lemma 3 (Generalized Hölder inequality)**.** *Let* $\alpha, \beta, \gamma \in \mathbb{R} \cup \{+\infty\}$ *such that* $\beta + \gamma \geq 0$ *and* $\frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{\alpha}$*. Then, for every* $a, b, c, d \geq 0$ *and* $\lambda \in [0, 1]$,

$$M_\alpha^\lambda(ac, bd) \leq M_\beta^\lambda(a, b) M_\gamma^\lambda(c, d).$$

Corollary 4 (Borell-Brascamp-Lieb inequality)**.** *Let* $\gamma \geq -\frac{1}{n}$*,* $\lambda \in [0, 1]$ *and* $f, g, h : \mathbb{R}^n \to [0, +\infty)$ *be measurable functions. If the inequality*

$$h((1-\lambda)x + \lambda y) \geq M_\gamma^\lambda(f(x), g(y))$$

holds for every $x \in \operatorname{supp}(f), y \in \operatorname{supp}(g)$*, then*

$$\int_{\mathbb{R}^n} h \geq M_{\frac{\gamma}{1+\gamma n}}^\lambda \left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g \right).$$

Corollary 4 follows from Theorem 1 by taking $\varphi(x, y) = (1-\lambda)x + \lambda y$, $x \in \operatorname{supp}(f), y \in \operatorname{supp}(g)$, and $\Phi(a, b) = M_{\frac{\gamma}{1+\gamma n}}^\lambda(a, b)$, $a, b \geq 0$. Indeed, using Lemma 3, one obtains that for every $x \in \operatorname{supp}(f), y \in \operatorname{supp}(g)$, and for every $\rho_1, \ldots, \rho_n, \eta_1, \ldots, \eta_n > 0$,

$$\begin{aligned}
h(\varphi(x, y)) \Pi_{k=1}^n \left(\frac{\partial \varphi}{\partial x_k} \rho_k + \frac{\partial \varphi}{\partial y_k} \eta_k \right) &= h((1-\lambda)x + \lambda y) \Pi_{k=1}^n ((1-\lambda)\rho_k + \lambda \eta_k) \\
&\geq M_\gamma^\lambda(f(x), g(y)) M_{\frac{1}{n}}^\lambda(\Pi_{k=1}^n \rho_k, \Pi_{k=1}^n \eta_k) \\
&\geq M_{\frac{\gamma}{1+\gamma n}}^\lambda(f(x) \Pi_{k=1}^n \rho_k, g(y) \Pi_{k=1}^n \eta_k) \\
&= \Phi(f(x) \Pi_{k=1}^n \rho_k, g(y) \Pi_{k=1}^n \eta_k).
\end{aligned}$$

Corollary 4 was independently proved by Borell (see [6, Theorem 3.1]), and by Brascamp and Lieb [8].

Another important consequence of the Borell-Brunn-Minkowski inequality is obtained when considering φ to be nonlinear. Let us denote for $\mathbf{p} = (p_1, \ldots, p_n) \in [-\infty, +\infty]^n$, $x = (x_1, \ldots, x_n) \in [0, +\infty]^n$ and $y = (y_1, \ldots, y_n) \in [0, +\infty]^n$,

$$M_{\mathbf{p}}^\lambda(x, y) = (M_{p_1}^\lambda(x_1, y_1), \ldots, M_{p_n}^\lambda(x_n, y_n)).$$

Corollary 5 (nonlinear extension of the Brunn-Minkowski inequality)**.** *Let* $\mathbf{p} = (p_1, \ldots, p_n) \in [0, 1]^n$*,* $\gamma \geq -(\sum_{i=1}^n p_i^{-1})^{-1}$*,* $\lambda \in [0, 1]$*, and* $f, g, h : [0, +\infty)^n \to [0, +\infty)$ *be measurable functions. If the inequality*

$$h(M_{\mathbf{p}}^\lambda(x, y)) \geq M_\gamma^\lambda(f(x), g(y))$$

holds for every $x \in \operatorname{supp}(f), y \in \operatorname{supp}(g)$*, then*

$$\int_{[0,+\infty)^n} h \geq M_{(\sum_{i=1}^n p_i^{-1} + \gamma^{-1})^{-1}}^\lambda \left(\int_{[0,+\infty)^n} f, \int_{[0,+\infty)^n} g \right).$$

Corollary 5 follows from Theorem 1 by taking $\varphi(x,y) = M_{\mathbf{p}}^{\lambda}(x,y)$, $x \in \mathrm{supp}(f), y \in \mathrm{supp}(g)$, and $\Phi(a,b) = M^{\lambda}_{(\sum_{i=1}^n p_i^{-1}+\gamma^{-1})^{-1}}(a,b)$, $a,b \geq 0$. Indeed, using Lemma 3, one obtains that for every $x \in \mathrm{supp}(f), y \in \mathrm{supp}(g)$, and for every $\rho_1, \dots, \rho_n, \eta_1, \dots, \eta_n > 0$,

$$\begin{aligned} h(\varphi(x,y))\Pi_{k=1}^n \left(\frac{\partial \varphi}{\partial x_k}\rho_k + \frac{\partial \varphi}{\partial y_k}\eta_k \right) &= h(M_{\mathbf{p}}^{\lambda}(x,y))\Pi_{k=1}^n M^{\lambda}_{\frac{p_k}{1-p_k}}(x_k^{1-p_k}, y_k^{1-p_k})M_1(x_k^{p_k-1}\rho_k, y_k^{p_k-1}\eta_k) \\ &\geq M_{\gamma}^{\lambda}(f(x),g(y))\Pi_{k=1}^n M^{\lambda}_{p_k}(\rho_k,\eta_k) \\ &\geq M_{\gamma}^{\lambda}(f(x),g(y)) M^{\lambda}_{(\sum_{i=1}^n p_i^{-1})^{-1}}(\Pi_{k=1}^n\rho_k, \Pi_{k=1}^n\eta_k) \\ &\geq M^{\lambda}_{(\sum_{i=1}^n p_i^{-1}+\gamma^{-1})^{-1}}(f(x)\Pi_{k=1}^n\rho_k, g(y)\Pi_{k=1}^n\eta_k) \\ &= \Phi(f(x)\Pi_{k=1}^n\rho_k, g(y)\Pi_{k=1}^n\eta_k). \end{aligned}$$

In the particular case where $\mathbf{p} = (0, \dots, 0)$, Corollary 5 was rediscovered by Ball [1]. In the general case, Corollary 5 was rediscovered by Uhrin [24].

Notice that the condition on p in Corollary 5 is less restrictive in dimension 1. It reads as follows:

Corollary 6 (nonlinear extension of the Brunn-Minkowski inequality on the line)**.** *Let $p \leq 1$, $\gamma \geq -p$, and $\lambda \in [0,1]$. Let $f,g,h : [0,+\infty) \to [0,+\infty)$ be measurable functions such that for every $x \in \mathrm{supp}(f), y \in \mathrm{supp}(g)$,*

$$h(M_p^{\lambda}(x,y)) \geq M_{\gamma}^{\lambda}(f(x),g(y)).$$

Then,

$$\int_0^{+\infty} h \geq M^{\lambda}_{\left(\frac{1}{p}+\frac{1}{\gamma}\right)^{-1}} \left(\int_0^{+\infty} f, \int_0^{+\infty} g \right).$$

A simple proof of Corollary 6 was recently given by Bobkov et al. [4].

In section 2, we present a simple proof of Theorem 1, based on mass transportation. In section 3, we discuss applications of the above inequalities to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang. We also prove an equivalence between the log-Brunn-Minkowski inequality and its possible extensions to convex measures (see section 3 for precise definitions).

2 A simple proof of the Borell-Brunn-Minkowski inequality

In this section, we present a simple proof of Theorem 1.

Proof of Theorem 1. The proof is done by induction on the dimension. To prove the theorem in dimension 1, we use a mass transportation argument.

Step 1 : (In dimension 1)
First let us see that if $\int f = 0$ or $\int g = 0$, then the result holds. Let us assume, without loss of generality, that $\int g = 0$. By taking $\rho = 1$, by letting η go to 0 and by using continuity and homogeneity of Φ in the condition (1), one obtains

$$h(\varphi(x,y))\frac{\partial \varphi}{\partial x} \geq \Phi(f(x),0) = f(x)\Phi(1,0).$$

It follows that, for fixed $y \in \mathrm{supp}(g)$,

$$\int h(z)\mathrm{d}z \geq \int_{\varphi(\mathrm{supp}(f),y)} h(z)\mathrm{d}z = \int_{\mathrm{supp}(f)} h(\varphi(x,y))\frac{\partial \varphi}{\partial x}\mathrm{d}x \geq \int f\Phi(1,0) = \Phi\left(\int f, \int g\right).$$

A similar argument shows that the result holds if $\int f = +\infty$ or $\int g = +\infty$. Thus we assume thereafter that $0 < \int f < +\infty$ and $0 < \int g < +\infty$.

Let us show that one may assume that $\int f = \int g = 1$. Let us define, for $x, y \in \mathbb{R}$ and $a, b \geq 0$,

$$\widetilde{f}(x) = f\left(\Phi\left(\int f, 0\right) x\right) \Phi(1,0), \quad \widetilde{g}(x) = g\left(\Phi\left(0, \int g\right) x\right) \Phi(0,1),$$

$$\widetilde{h}(x) = h\left(\Phi\left(\int f, \int g\right) x\right),$$

$$\widetilde{\varphi}(x,y) = \frac{\varphi(\Phi(\int f, 0)x, \Phi(0, \int g)y)}{\Phi(\int f, \int g)}, \quad \widetilde{\Phi}(a,b) = \Phi\left(a \frac{\int f}{\Phi(\int f, \int g)}, b \frac{\int g}{\Phi(\int f, \int g)}\right).$$

Let $x \in \mathrm{supp}(\widetilde{f})$, $y \in \mathrm{supp}(\widetilde{g})$, and let $\widetilde{\rho}, \widetilde{\eta} > 0$. One has,

$$\begin{aligned} \widetilde{h}(\widetilde{\varphi}(x,y)) \left(\frac{\partial \widetilde{\varphi}}{\partial x} \widetilde{\rho} + \frac{\partial \widetilde{\varphi}}{\partial y} \widetilde{\eta}\right) &\geq \Phi\left(f(\Phi(\int f, 0)x) \frac{\Phi(\int f, 0)}{\Phi(\int f, \int g)} \widetilde{\rho}, g(\Phi(0, \int g)y) \frac{\Phi(0, \int g)}{\Phi(\int f, \int g)} \widetilde{\eta}\right) \\ &= \widetilde{\Phi}(\widetilde{f}(x)\widetilde{\rho}, \widetilde{g}(y)\widetilde{\eta}). \end{aligned}$$

Notice that the functions $\widetilde{\varphi}$ and $\widetilde{\Phi}$ satisfy the same assumptions as the functions φ and Φ respectively, and that $\int \widetilde{f} = \int \widetilde{g} = 1$. If the result holds for functions of integral one, then

$$\int \widetilde{h}(w) \mathrm{d}w \geq \widetilde{\Phi}(1,1) = 1.$$

The change of variable $w = z/\Phi(\int f, \int g)$ leads us to

$$\int h(z) \mathrm{d}z \geq \Phi\left(\int f, \int g\right).$$

Assume now that $\int f = \int g = 1$. By standard approximation, one may assume that f and g are compactly supported positive Lipschitz functions (relying on the fact that Φ is continuous and increasing in each coordinate, compare with [2, page 343]). Thus there exists a non-decreasing map $T : \mathrm{supp}(f) \to \mathrm{supp}(g)$ such that for every $x \in \mathrm{supp}(f)$,

$$f(x) = g(T(x))T'(x),$$

see e.g. [3], [25]. Since T is non-decreasing and $\partial\varphi/\partial x, \partial\varphi/\partial y > 0$, the function $\Theta : \mathrm{supp}(f) \to \varphi(\mathrm{supp}(f), T(\mathrm{supp}(f)))$ defined by $\Theta(x) = \varphi(x, T(x))$ is bijective. Hence the change of variable $z = \Theta(x)$ is admissible and one has,

$$\begin{aligned} \int h(z) \mathrm{d}z \geq \int_{\mathrm{supp}(f)} h(\varphi(x, T(x))) \left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} T'(x)\right) \mathrm{d}x &\geq \int_{\mathrm{supp}(f)} \Phi(f(x), g(T(x))T'(x)) \mathrm{d}x \\ &= \int \Phi(f(x), f(x)) \mathrm{d}x. \end{aligned}$$

Using homogeneity of Φ, one deduces that

$$\int h \geq \Phi(1,1) \int f(x) \mathrm{d}x = \Phi\left(\int f, \int g\right).$$

Step 2 : (Tensorization)

Let n be a positive integer and assume that Theorem 1 holds in $\mathbb{R}^n$. Let f, g, h, φ, Φ satisfying the assumptions of Theorem 1 in $\mathbb{R}^{n+1}$. Recall that the inequality

$$h(\varphi(x,y)) \Pi_{k=1}^{n+1} \left(\frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k\right) \geq \Phi(f(x) \Pi_{k=1}^{n+1} \rho_k, g(y) \Pi_{k=1}^{n+1} \eta_k), \tag{2}$$

holds for every $x \in \mathrm{supp}(f), y \in \mathrm{supp}(g)$, and for every $\rho_1, \ldots, \rho_{n+1}, \eta_1, \ldots, \eta_{n+1} > 0$. Let us define, for $x_{n+1}, y_{n+1}, z_{n+1} \in \mathbb{R}$,

$$F(x_{n+1}) = \int_{\mathbb{R}^n} f(x, x_{n+1})\mathrm{d}x, \quad G(y_{n+1}) = \int_{\mathbb{R}^n} g(x, g_{n+1})\mathrm{d}x, \quad H(z_{n+1}) = \int_{\mathbb{R}^n} h(x, z_{n+1})\mathrm{d}x.$$

Since $\int f > 0, \int g > 0$, the support of F and the support of G are nonempty. Let $x_{n+1} \in \mathrm{supp}(F), y_{n+1} \in \mathrm{supp}(G)$, and let $\rho_{n+1}, \eta_{n+1} > 0$. Let us define, for $x, y, z \in \mathbb{R}^n$,

$$f_{x_{n+1}}(x) = f(x, x_{n+1})\rho_{n+1}, \quad g_{y_{n+1}}(y) = g(y, y_{n+1})\eta_{n+1}, \quad \overline{\varphi}(x, y) = (\varphi_1(x_1, y_1), \ldots, \varphi_n(x_n, y_n)),$$

$$h_{\varphi_{n+1}}(z) = h(z, \varphi_{n+1}(x_{n+1}, y_{n+1})) \left(\frac{\partial \varphi_{n+1}}{\partial x_{n+1}} \rho_{n+1} + \frac{\partial \varphi_{n+1}}{\partial y_{n+1}} \eta_{n+1} \right).$$

Let $x \in \mathrm{supp}(f_{x_{n+1}}), y \in \mathrm{supp}(g_{y_{n+1}})$, and let $\rho_1, \ldots, \rho_n, \eta_1, \ldots, \eta_n > 0$. One has

$$\begin{aligned} h_{\varphi_{n+1}}(\overline{\varphi}(x, y))\Pi_{k=1}^n \left(\frac{\partial \overline{\varphi_k}}{\partial x_k} \rho_k + \frac{\partial \overline{\varphi_k}}{\partial y_k} \eta_k \right) &= h(\varphi(x, x_{n+1}, y, y_{n+1}))\Pi_{k=1}^{n+1} \left(\frac{\partial \varphi_k}{\partial x_k} \rho_k + \frac{\partial \varphi_k}{\partial y_k} \eta_k \right) \\ &\geq \Phi(f(x, x_{n+1})\Pi_{k=1}^{n+1}\rho_k, g(y, y_{n+1})\Pi_{k=1}^{n+1}\eta_k) \\ &= \Phi(f_{x_{n+1}}(x)\Pi_{k=1}^n\rho_k, g_{y_{n+1}}(y)\Pi_{k=1}^n\eta_k), \end{aligned}$$

where the inequality follows from inequality (2). Hence, applying Theorem 1 in dimension n, one has

$$\int_{\mathbb{R}^n} h_{\varphi_{n+1}}(x)\mathrm{d}x \geq \Phi \left(\int_{\mathbb{R}^n} f_{x_{n+1}}(x)\mathrm{d}x, \int_{\mathbb{R}^n} g_{y_{n+1}}(x)\mathrm{d}x \right).$$

This yields that for every $x_{n+1} \in \mathrm{supp}(F), y_{n+1} \in \mathrm{supp}(G)$, and for every $\rho_{n+1}, \eta_{n+1} > 0$,

$$H(\varphi_{n+1}(x_{n+1}, y_{n+1})) \left(\frac{\partial \varphi_{n+1}}{\partial x_{n+1}} \rho_{n+1} + \frac{\partial \varphi_{n+1}}{\partial y_{n+1}} \eta_{n+1} \right) \geq \Phi(F(x_{n+1}), G(y_{n+1})).$$

Hence, applying Theorem 1 in dimension 1, one has

$$\int_{\mathbb{R}} H(x)\mathrm{d}x \geq \Phi \left(\int_{\mathbb{R}} F(x)\mathrm{d}x, \int_{\mathbb{R}} G(x)\mathrm{d}x \right).$$

This yields the desired inequality. □

3 Applications to the log-Brunn-Minkowski inequality

In this section, we discuss applications of the above inequalities to the log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang [7].

Recall that a *convex body* in $\mathbb{R}^n$ is a compact convex subset of $\mathbb{R}^n$ with nonempty interior. Böröczky et al. conjectured the following inequality.

Conjecture 7 (log-Brunn-Minkowski inequality)**.** *Let K, L be symmetric convex bodies in $\mathbb{R}^n$ and let $\lambda \in [0, 1]$. Then,*

$$|(1 - \lambda) \cdot K \oplus_0 \lambda \cdot L| \geq |K|^{1-\lambda}|L|^\lambda.$$

Here,

$$(1 - \lambda) \cdot K \oplus_0 \lambda \cdot L = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_K(u)^{1-\lambda}h_L(u)^\lambda, \text{ for all } u \in S^{n-1}\},$$

where S^{n-1} denotes the n-dimensional Euclidean unit sphere, h_K denotes the support function of K, defined by $h_K(u) = \max_{x \in K}\langle x, u \rangle$, and $|\cdot|$ stands for Lebesgue measure.

Böröczky et al. [7] proved that Conjecture 7 holds in the plane. Using Corollary 5 with $\mathbf{p} = (0, \ldots, 0)$, Saroglou [21] proved that Conjecture 7 holds for unconditional convex bodies in $\mathbb{R}^n$ (a set $K \subset \mathbb{R}^n$ is *unconditional* if for every $(x_1, \dots, x_n) \in K$ and for every $(\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$, one has $(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \in K$).

Recall that a measure μ is *s-concave*, $s \in [-\infty, +\infty]$, if the inequality

$$\mu((1-\lambda)A + \lambda B) \geq M_s^\lambda(\mu(A), \mu(B))$$

holds for all compact sets $A, B \subset \mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$ and for every $\lambda \in [0, 1]$ (see [5], [6]). The 0-concave measures are also called *log-concave measures*, and the $-\infty$-concave measures are also called *convex measures*. A function $f : \mathbb{R}^n \to [0, +\infty)$ is α-concave, $\alpha \in [-\infty, +\infty]$, if the inequality

$$f((1-\lambda)x + \lambda y) \geq M_\alpha^\lambda(f(x), f(y))$$

holds for every $x, y \in \mathbb{R}^n$ such that $f(x)f(y) > 0$ and for every $\lambda \in [0, 1]$.

Saroglou [22] recently proved that if the log-Brunn-Minkowski inequality holds, then the inequality

$$\mu((1-\lambda) \cdot K \oplus_0 \lambda \cdot L) \geq \mu(K)^{1-\lambda} \mu(L)^\lambda$$

holds for every symmetric log-concave measure μ, for all symmetric convex bodies K, L in $\mathbb{R}^n$ and for every $\lambda \in [0, 1]$.

An extension of the log-Brunn-Minkowski inequality for convex measures was proposed by the author in [19], and reads as follows:

Conjecture 8. *Let $p \in [0, 1]$. Let μ be a symmetric measure in $\mathbb{R}^n$ that has an α-concave density function, with $\alpha \geq -\frac{p}{n}$. Then for every symmetric convex body K, L in $\mathbb{R}^n$ and for every $\lambda \in [0, 1]$,*

$$\mu((1-\lambda) \cdot K \oplus_p \lambda \cdot L) \geq M^\lambda_{\left(\frac{n}{p}+\frac{1}{\alpha}\right)^{-1}}(\mu(K), \mu(L)). \tag{3}$$

Here,

$$(1-\lambda) \cdot K \oplus_p \lambda \cdot L = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq M_p^\lambda(h_K(u), h_L(u)), \text{ for all } u \in S^{n-1}\}.$$

In Conjecture 8, if α or p is equal to 0, then $(n/p + 1/\alpha)^{-1}$ is defined by continuity and is equal to 0. Notice that Conjecture 7 is a particular case of Conjecture 8 when taking μ to be Lebesgue measure and $p = 0$.

By using Corollary 6, we will prove that Conjecture 7 implies Conjecture 8, when $\alpha \leq 1$, generalizing Saroglou's result discussed earlier.

Theorem 9. *If the log-Brunn-Minkowski inequality holds, then the inequality*

$$\mu((1-\lambda) \cdot K \oplus_p \lambda \cdot L) \geq M^\lambda_{\left(\frac{n}{p}+\frac{1}{\alpha}\right)^{-1}}(\mu(K), \mu(L))$$

holds for every $p \in [0, 1]$, for every symmetric measure μ in $\mathbb{R}^n$ that has an α-concave density function, with $1 \geq \alpha \geq -\frac{p}{n}$, for every symmetric convex body K, L in $\mathbb{R}^n$ and for every $\lambda \in [0, 1]$.

Proof. Let K_0, K_1 be symmetric convex bodies in $\mathbb{R}^n$ and let $\lambda \in (0, 1)$. Let us denote $K_\lambda = (1-\lambda) \cdot K_0 \oplus_p \lambda \cdot K_1$ and let us denote by ψ the density function of μ. Let us define, for $t > 0$, $h(t) = |K_\lambda \cap \{\psi \geq t\}|$, $f(t) = |K_0 \cap \{\psi \geq t\}|$ and $g(t) = |K_1 \cap \{\psi \geq t\}|$. Notice that

$$\mu(K_\lambda) = \int_{K_\lambda} \psi(x) \mathrm{d}x = \int_{K_\lambda} \int_0^{\psi(x)} \mathrm{d}t \mathrm{d}x = \int_0^{+\infty} |K_\lambda \cap \{\psi \geq t\}| = \int_0^{+\infty} h(t) \mathrm{d}t.$$

Similarly, one has

$$\mu(K_0) = \int_0^{+\infty} f(t) \mathrm{d}t, \qquad \mu(K_1) = \int_0^{+\infty} g(t) \mathrm{d}t.$$

Let $t, s > 0$ such that the sets $\{\psi \geq t\}$ and $\{\psi \geq s\}$ are nonempty. Let us denote $L_0 = \{\psi \geq t\}$, $L_1 = \{\psi \geq s\}$ and $L_\lambda = \{\psi \geq M_\alpha^\lambda(t,s)\}$. If $x \in L_0$ and $y \in L_1$, then $\psi((1-\lambda)x + \lambda y) \geq M_\alpha^\lambda(\psi(x), \psi(y)) \geq M_\alpha^\lambda(t,s)$. Hence,

$$L_\lambda \supset (1-\lambda)L_0 + \lambda L_1 \supset (1-\lambda) \cdot L_0 \oplus_p \lambda \cdot L_1,$$

the last inclusion following from the fact that $p \leq 1$. We deduce that

$$K_\lambda \cap L_\lambda \supset ((1-\lambda)\cdot K_0 \oplus_p \lambda \cdot K_1) \cap ((1-\lambda)\cdot L_0 \oplus_p \lambda \cdot L_1) \supset (1-\lambda)\cdot(K_0 \cap L_0) \oplus_p \lambda \cdot (K_1 \cap L_1).$$

Hence,

$$h(M_\alpha^\lambda(t,s)) = |K_\lambda \cap L_\lambda| \geq |(1-\lambda)\cdot(K_0 \cap L_0) \oplus_p \lambda \cdot (K_1 \cap L_1)| \geq M_{\frac{p}{n}}^\lambda(f(t), g(s)),$$

the last inequality is valid for $p \geq 0$ and follows from the log-Brunn-Minkowski inequality by using homogeneity of Lebesgue measure (see [7, beginning of section 3]). Thus we may apply Corollary 6 to conclude that

$$\mu(K_\lambda) = \int_0^{+\infty} h \geq M^\lambda_{\left(\frac{n}{p}+\frac{1}{\alpha}\right)^{-1}}\left(\int_0^{+\infty} f, \int_0^{+\infty} g\right) = M^\lambda_{\left(\frac{n}{p}+\frac{1}{\alpha}\right)^{-1}}(\mu(K_0), \mu(K_1)).$$

□

Since the log-Brunn-Minkowski inequality holds true in the plane, we deduce that Conjecture 8 holds true in the plane (with the restriction $\alpha \leq 1$). Notice that Conjecture 8 holds true in the unconditional case as a consequence of Corollary 5 (see [19]).

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