ON THE STABILITY OF BRUNN-MINKOWSKI TYPE INEQUALITIES

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ABSTRACT. We establish the stability near a Euclidean ball of two conjectured inequalities: the dimensional Brunn-Minkowski inequality for radially symmetric log-concave measures in $\mathbb{R}^n$, and of the log-Brunn-Minkowski inequality.

1. INTRODUCTION

The classical Brunn-Minkowski inequality states that for $\lambda \in [0,1]$ and for Borel measurable sets A and B in $\mathbb{R}^n$, such that $(1-\lambda)A + \lambda B$ is measurable as well,

$$|\lambda A + (1-\lambda)B|^{\frac{1}{n}} \geq \lambda |A|^{\frac{1}{n}} + (1-\lambda)|B|^{\frac{1}{n}}. \tag{1}$$

Here $|\cdot|$ denotes the Lebesgue measure, the addition between sets is the standard vector addition, and multiplication of sets by non-negative reals is the usual dilation.

This inequality has found many important applications in Geometry and Analysis (see *e.g.* Gardner [16] for an exhaustive survey on this subject). For example, the classical isoperimetric inequality can be deduced in a few lines from (1). Also, Maurey [29] deduced from this inequality the Poincaré inequality for the Gaussian measure and Gaussian concentration properties. Based on Maurey's results, Bobkov and Ledoux proved that the Brunn-Minkowski inequality implies Brascamp-Lieb and log-Sobolev inequalities [3]; they also deduced sharp Sobolev and Gagliardo-Nirenberg inequalities [4]. A different argument was developed by the first named author in [11] to deduce Poincaré type inequalities on the boundary of convex bodies from the Brunn-Minkowski inequality.

Recall that a convex body is a convex compact set with non-empty interior. The family of convex bodies of $\mathbb{R}^n$ will be denoted by $\mathcal{K}^n$. For the theory of convex bodies we refer the reader to the books by Ball [1], Bonnesen, Fenchel [5], Koldobsky [20], Milman and Schechtman [30], Schneider [38] and others. A measure γ on $\mathbb{R}^n$ is called log-concave if for any pair of sets A and B and for any scalar $\lambda \in [0,1]$,

$$\gamma(\lambda A + (1-\lambda)B) \geq \gamma(A)^{\lambda}\gamma(B)^{1-\lambda}. \tag{2}$$

Borell showed [6] that a measure is log-concave if it has a density (with respect to the Lebesgue measure) which is log-concave (see also Prékopa [34], Leindler [24]). In particular, the Lebesgue measure on $\mathbb{R}^n$ is log-concave:

$$|\lambda A + (1-\lambda)B| \geq |A|^{\lambda}|B|^{1-\lambda}. \tag{3}$$

Inequality (1) implies (3) by the arithmetic-geometric mean inequality. Conversely, a simple argument based on the homogeneity of Lebesgue measure shows that (3) implies (1) (see, for example, [16]). In general, a property analogous to (1) may not hold for log-concave measures which are not homogeneous. The transposition of (1) to a measure γ,

$$\gamma(\lambda A + (1-\lambda)B)^{\frac{1}{n}} \geq \lambda\gamma(A)^{\frac{1}{n}} + (1-\lambda)\gamma(B)^{\frac{1}{n}}, \quad \forall\, \lambda \in [0,1], \tag{4}$$

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as A and B vary in some class of sets, will be called a **dimensional Brunn-Minkowski inequality**. If γ is the Gaussian measure, $A = \{p\}$, $p \in \mathbb{R}^n$, and B is measurable set with positive measure, then the set $A + B$ is the translate of B by p. Hence, letting $|p| \to \infty$, and keeping B fixed, (4) fails. Moreover, Nayar and Tkocz [32] constructed an example in which (4) fails for the Gaussian measure while both A and B contain the origin. Gardner and Zvavitch [17] proved that, for the Gaussian measure, (4) holds if the sets A and B are convex symmetric dilates of each other. They also proposed a conjecture for the Gaussian measure, that we state it in a more general form.

Conjecture 1.1 (Gardner, Zvavitch – generalized). *Let $n \geq 2$. Let γ be a log-concave symmetric measure (i.e. $\gamma(A) = \gamma(-A)$ for every measurable set A) on $\mathbb{R}^n$. Let K and L be symmetric convex bodies in $\mathbb{R}^n$. Then*

$$\gamma(\lambda K + (1-\lambda)L)^{\frac{1}{n}} \geq \lambda\gamma(K)^{\frac{1}{n}} + (1-\lambda)\gamma(L)^{\frac{1}{n}}. \tag{5}$$

Next, we pass to describe the log-Brunn-Minkowski inequality. For a scalar $\lambda \in [0, 1]$ and for convex bodies K and L containing the origin in their interior, with support functions h_K and h_L, respectively (see section 2 for the definition), define their geometric average as follows:

$$K^\lambda L^{1-\lambda} := \{x \in \mathbb{R}^n \,:\, \langle x, u\rangle \leq h_K^\lambda(u) h_L^{1-\lambda}(u) \ \forall u \in \mathbb{S}^{n-1}\}, \tag{6}$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$. This set is again a convex body, whose support function is, in general, smaller than the geometric mean of the support functions of K and L. The following is widely known as log-Brunn-Minkowski conjecture (see [7]).

Conjecture 1.2 (Böröczky, Lutwak, Yang, Zhang). *Let $n \geq 2$ be an integer. Let K and L be symmetric convex bodies in $\mathbb{R}^n$. Then*

$$|K^\lambda L^{1-\lambda}| \geq |K|^\lambda |L|^{1-\lambda}. \tag{7}$$

Important applications and motivations for Conjecture 1.2 can be found in [8], [9].

It is not difficult to see that the condition of symmetry is necessary (see [7] or Remark 1.5 below). Böröczky, Lutwak, Yang and Zhang showed that this conjecture holds for $n = 2$. Saroglou [36] and Cordero, Fradelizi, Maurey [15] proved that (7) is true when the sets K and L are unconditional (i.e. they are symmetric with respect to every coordinate hyperplane). Rotem [35] showed that log-Brunn-Minkowski conjecture holds for complex convex bodies. Saroglou showed [37] that the validity of Conjecture 1.2 would imply the same statement for every log-concave symmetric measure γ on $\mathbb{R}^n$: for every symmetric $K, L \in \mathcal{K}^n$ and for every $\lambda \in [0, 1]$,

$$\gamma(K^\lambda L^{1-\lambda}) \geq \gamma(K)^\lambda \gamma(L)^{1-\lambda}. \tag{8}$$

Note that the straightforward inclusion

$$K^\lambda L^{1-\lambda} \subset \lambda K + (1-\lambda)L$$

tells us that (8) is stronger than (2), for every measure.

In [27] the second and third named authors, Nayar and Zvavitch showed that (8) implies (5) for every ray-decreasing measure γ on $\mathbb{R}^n$ and for every pair of convex sets K and L. Therefore, Conjecture 1.1 holds on the plane and for unconditional sets.

The main results of this paper are the two theorems below.

Theorem 1.3 (The dimensional Brunn-Minkowski inequality near a ball). *Let γ be a rotation invariant log-concave measure on $\mathbb{R}^n$. Let $R \in (0, \infty)$. Let $\psi \in C^2(\mathbb{S}^{n-1})$. Then there* *exists a sufficiently small $a > 0$ such that for every $\epsilon_1, \epsilon_2 \in (0, a)$ and for every $\lambda \in [0, 1]$, one has*

$$\gamma(\lambda K_1 + (1-\lambda)K_2)^{\frac{1}{n}} \geq \lambda\gamma(K_1)^{\frac{1}{n}} + (1-\lambda)\gamma(K_2)^{\frac{1}{n}},$$

where K_1 is the convex set with the support function $h_1 = R + \epsilon_1\psi$ and K_2 is the convex set with the support function $h_2 = R + \epsilon_2\psi$.

Theorem 1.4 (The log-Brunn-Minkowski inequality near a ball)**.** *Let γ be a rotation invariant log-concave measure on $\mathbb{R}^n$. Let $R \in (0, \infty)$. Let $\varphi \in C^2(\mathbb{S}^{n-1})$ be* ***even*** *and strictly positive. Then there exists a sufficiently small $a > 0$ such that for every $\epsilon_1, \epsilon_2 \in (0, a)$ and for every $\lambda \in [0, 1]$, one has*

$$\gamma(K_1^{\lambda}K_2^{1-\lambda}) \geq \gamma(K_1)^{\lambda}\gamma(K_2)^{1-\lambda},$$

where K_1 is the convex set with the support function $h_1 = R\varphi^{\epsilon_1}$ and K_2 is the convex set with the support function $h_2 = R\varphi^{\epsilon_2}$.

Theorem 1.4 can be used to obtain a local uniqueness result for log-Minkowski problem (see Böröczky, Lutwak, Yang, Zhang [7], [8] and the references therein), and the corresponding investigation shall be carried out in a separate manuscript.

Remark 1.5. *Theorems 1.3 and 1.4 indicate a difference between the local behaviors of the dimensional Brunn-Minkowski inequality and the log-Brunn-Minkowski inequality. Indeed,* (7) *fails for the simplest possible odd perturbation: the shift (which is equivalent to chosing φ as the restriction of a linear function to $\mathbb{S}^{n-1}$). In contrast, by Theorem 1.3 the Brunn-Minkowski inequality holds for radially symmetric log-concave measures when K and L are perturbations, non necessarily even, of RB_2^n.*

This paper is structured as follows. Section 2 contains some preliminary material for the subsequent part of the paper. In Section 3 we discuss the relations between the dimensional Brunn-Minkowski inequality and the log-Brunn-Minkowski inequality and their infinitesimal forms. Theorems 1.3 and 1.4 are proved in Sections 4 and 5, respectively. Finally, we provide some technical details in the Section 6.

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2. Preliminaries

We work in the $n-$dimensional Euclidean space $\mathbb{R}^n$ with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$. We set $B_2^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$, to denote the unit ball and the unit sphere, respectively. We shall denote the Lebesgue measure (the *volume*) in $\mathbb{R}^n$ by $|\cdot|$.

We say that a set $A \subset \mathbb{R}^n$ is symmetric if for every $x \in A$ one has $-x \in A$. All measures under consideration will be tacitly assumed to be Radon measures, and all sets will be assumed to be measurable. A measure γ on $\mathbb{R}^n$ is called symmetric if for every set $S \subset \mathbb{R}^n$, $\gamma(S) = \gamma(-S)$. If the measure has a density then it is symmetric whenever the density is an even function.

A measure γ on $\mathbb{R}^n$ is said to be rotation invariant if for every set $A \subset \mathbb{R}^n$, and for every rotation T, $\gamma(A) = \gamma(TA)$. If a rotation invariant measure γ has a density F, we may write F in the form:

$$F(x) = f(|x|),$$

for a suitable $f : [0, \infty) \to [0, \infty)$.

For $K \in \mathcal{K}^n$, the *support function* of K, $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$, is defined as

$$h_K(u) = \sup_{x \in K} \langle x, u \rangle.$$

By the geometric viewpoint, $h_K(u)$ represents the (signed) distance from the origin of the supporting hyperplane to K with outer unit normal u. We shall use the notation $H_K(x)$ for the 1-homogenous extension of h_K, that is,

$$H_K(x) = \begin{cases} |x| \, h_K\left(\dfrac{x}{|x|}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The function H_K is convex in $\mathbb{R}^n$, for every $K \in \mathcal{K}^n$. Vice versa, for every continuous 1-homogeneous convex function H on $\mathbb{R}^n$, there exists a unique convex body K such that $H = H_K$.

Note that $K \in \mathcal{K}^n$ contains the origin (resp., in its interior) if and only if $h_K \geq 0$ (resp. $h_K > 0$) on $\mathbb{S}^{n-1}$. For convex bodies K and L, and for α, $\beta \geq 0$, we have:

$$h_{\alpha K + \beta L}(u) = \alpha h_K(u) + \beta h_L(u). \tag{9}$$

We say that a convex body K is $C^{2,+}$ if ∂K is of class C^2 and the Gauss curvature is strictly positive at every $x \in \partial K$. In particular, if K is $C^{2,+}$ then it admits outer unit normal $\nu_K(x)$ at every boundary point x. Recall that the Gauss map $\nu_K : \partial K \to \mathbb{S}^{n-1}$ is the map assigning the unit normal to each point of ∂K.

$C^{2,+}$ convex bodies can be characterized through their support function. We recall that an orthonormal frame on the sphere is a map which associates a collection of $n-1$ orthonormal vectors to every point of $\mathbb{S}^{n-1}$. Let $\psi \in C^2(\mathbb{S}^{n-1})$. We denote by $\psi_i(u)$ and $\psi_{ij}(u)$, $i, j \in \{1, \dots, n-1\}$, the first and second covariant derivatives of ψ at $u \in \mathbb{S}^{n-1}$, with respect to a fixed local orthonormal frame on an open subset of $\mathbb{S}^{n-1}$. We define the matrix

$$Q(\psi; u) = (q_{ij})_{i,j=1,\dots,n-1} = (\psi_{ij}(u) + \psi(u)\delta_{ij})_{i,j=1,\dots,n-1}, \tag{10}$$

where the δ_{ij}'s are the usual Kronecker symbols. On an occasion, instead of $Q(\psi; u)$ we write $Q(\psi)$. Note that $Q(\psi; u)$ is symmetric by standard properties of covariant derivatives. The meaning of this matrix becomes particularly important when ψ is the support function of a convex body K. In this case we shall call it *curvature matrix* of K (see the following Remark 2.2). The proof of the following proposition can be deduced from Schneider [38, Section 2.5].

Proposition 2.1. *Let $K \in \mathcal{K}^n$ and let h be its support function. Then K is of class $C^{2,+}$ if and only if $h \in C^2(\mathbb{S}^{n-1})$ and*

$$Q(h; u) > 0 \quad \forall \, u \in \mathbb{S}^{n-1}.$$

In view of the previous results it is convenient to introduce the following set of functions

$$C^{2,+}(\mathbb{S}^{n-1}) = \{h \in C^2(\mathbb{S}^{n-1}) \, : \, Q(h; u) > 0 \, \forall \, u \in \mathbb{S}^{n-1}\}.$$

Hence $C^{2,+}(\mathbb{S}^{n-1})$ is the set of support functions of convex bodies of class $C^{2,+}$.

Remark 2.2. *Let K be a $C^{2,+}$ convex body. Then $\nu_K : \partial K \to \mathbb{S}^{n-1}$ is a diffeomorphism. The matrix $Q(h;u)$ represents the inverse of the Weingarten map at $x = \nu_K^{-1}(u)$, and its eigenvalues are the principal radii of curvature of ∂K at x. Consequently we have*

$$\det(Q(h;u)) = \frac{1}{G(x)}$$

where G denotes the Gauss curvature.

Let K be a $C^{2,+}$ convex body, with support function h_K and its homogenous extension H_K. H_K is of class $C^1(\mathbb{R}^n\setminus\{0\})$. By ∇H_K we denote its gradient with respect to Cartesian coordinates. The following useful relation holds: for every $u \in \mathbb{S}^{n-1}$, $\nabla H_K(u)$ is the (unique) point on ∂K where the outer unit normal is u:

$$\nabla H_K(u) = \nu_K^{-1}(u) \quad \forall\, u \in \mathbb{S}^{n-1}.$$

In other words,

$$\langle \nabla H_K(u), \nu_K(u)\rangle = H_K(u) \quad \forall\, u \in \mathbb{S}^{n-1}.$$

Remark 2.3. *Let $\psi \in C^1(\mathbb{S}^{n-1})$. The notation $\nabla_\sigma \psi$ stands for the* spherical gradient *of ψ, i.e. the vector $(\psi_1, \dots, \psi_{n-1})$, where ψ_i are the covariant derivatives of ψ with respect to the i-th element of a fixed orthonormal system on $\mathbb{S}^{n-1}$. Let Φ be the 1-homogeneous extension of ψ to $\mathbb{R}^n$. Then we have*

$$|\nabla \Phi(u)|^2 = \psi^2(u) + |\nabla_\sigma \psi(u)|^2 \tag{11}$$

for every $u \in \mathbb{S}^{n-1}$.

3. Infinitesimal versions of inequalities.

We denote the family of centrally symmetric convex bodies by $\mathcal{K}_s^n$. The notation $C_e^{2,+}(\mathbb{S}^{n-1})$ will stand for the set of support functions of centrally symmetric $C^{2,+}$ convex bodies, i.e. functions from $C^{2,+}(\mathbb{S}^{n-1})$ which are additionally even.

Let h be the support function of a $C^{2,+}$ convex body K, and let $\psi \in C^2(\mathbb{S}^{n-1})$; then, by Proposition 2.1,

$$h_s := h + s\psi \in C^{2,+}(\mathbb{S}^{n-1}) \tag{12}$$

if s is sufficiently small, say $|s| \le a$ for some appropriate $a > 0$. Hence for every s in this range there exists a unique $C^{2,+}$ convex body K_s with the support function h_s. For an interval I, we define the one-parameter family of convex bodies:

$$\mathbf{K}(h, \psi, I) := \{K_s \,:\, h_{K_s} = h + s\psi,\ s \in I\}.$$

Lemma 3.1. *Assume that γ is a symmetric log-concave measure with continuously differentiable density. Conjecture 1.1 holds for γ if and only if for every one-parameter family $\boldsymbol{K}(h, \psi, I)$, with even h and ψ,*

$$\frac{d^2}{ds^2}\left[\gamma(K_s)\right]\bigg|_{s=0} \cdot \gamma(K_0) \le \frac{n-1}{n}\left(\frac{d}{ds}\left[\gamma(K_s)\right]\bigg|_{s=0}\right)^2. \tag{13}$$

In particular, if (13) holds for K_s in a fixed family $\boldsymbol{K}(h, \psi, I)$, then Conjecture 1.1 holds for all sets K_s in that family.

Proof. Assume first that γ satisfies (5) on the system $\mathbf{K}(h, \psi, I)$. Then the equality $h_{K_s} = h + s\psi$, $s \in I$, and the linearity of support function with respect to Minkowski addition, imply that for every $s, t \in I$ and for every $\lambda \in [0, 1]$

$$K_{\lambda s + (1-\lambda)t} = \lambda K_s + (1-\lambda) K_t.$$

By (5),

$$\gamma(K_{\lambda s+(1-\lambda)t})^{\frac{1}{n}} = \gamma(\lambda K_s + (1-\lambda)K_t)^{\frac{1}{n}} \geq \lambda\gamma(K_s)^{\frac{1}{n}} + (1-\lambda)\gamma(K_t)^{\frac{1}{n}},$$

which means that the function $\gamma(K_s)^{\frac{1}{n}}$ is concave on I. Inequality (13) follows.

Conversely, suppose that for every system $\mathbf{K}(h,\psi,I)$ the function $\gamma(K_s)^{\frac{1}{n}}$ has non-positive second derivative at 0, i.e. (13) holds. We observe that this implies concavity of $\gamma(K_s)^{\frac{1}{n}}$ on the entire interval I. Indeed, given s_0 in the interior of I, consider $\tilde{h} = h + s_0\psi$, and define a new system $\tilde{\mathbf{K}}(\tilde{h},\psi,J)$, where J is a new interval such that $\tilde{h} + s\psi = h + (s+s_0)\psi \in C^{2,+}$ for every $s \in J$. Then the second derivative of $\gamma(K_s)^{\frac{1}{n}}$ at $s = s_0$ is negative, as it is equal to the second derivative of $\gamma(\tilde{K}_s)^{\frac{1}{n}}$ at $s = 0$. On the other hand, the concavity $\gamma(K_s)^{\frac{1}{n}}$ on the family $\mathbf{K}(h,\psi,I)$ is equivalent to the validity of (5) on this family. □

A similar approach can be used for the log-Brunn-Minkowski inequality. In order to do this we introduce a corresponding type of one-parameter families of convex bodies. In this case, additive perturbations are replaced by multiplicative perturbations.

Let $h \in C^{2,+}(\mathbb{S}^{n-1})$ and $\varphi \in C^2(\mathbb{S}^{n-1})$, with $\varphi > 0$ on $\mathbb{S}^{n-1}$. Then there exists $a > 0$ such that

$$h_s := h\,\varphi^s \in C^{2,+}(\mathbb{S}^{n-1}) \quad \forall\, s \in [-a,a].$$

In particular, by Proposition 2.1, for every $s \in [-a,a]$ there exists a $C^{2,+}$ convex body Q_s whose support function is h_s.

We introduce the corresponding 1-dimensional systems.

$$\mathbf{Q}(h,\varphi,I) := \{Q_s \in \mathcal{K}^n \,:\, h_{Q_s} = h\varphi^s,\ s \in I\}.$$

Lemma 3.2. *Let γ be a symmetric log-concave measure with continuously differentiable density. Assume that Conjecture 1.2 holds for a measure γ, i.e.* (8) *is valid for every pair of symmetric convex sets K and L and for every $\lambda \in [0,1]$. Then for every one-parameter family $Q_s \in \boldsymbol{Q}(h,\varphi,I)$, with h and φ even,*

$$\frac{d^2}{ds^2}\log(\gamma(Q_s))\bigg|_{s=0} \leq 0. \tag{14}$$

The converse is true locally: if (14) holds for all Q_s in a fixed family $\boldsymbol{Q}(h,\varphi,I)$, then Conjecture 1.2 holds for all sets Q_s in $\boldsymbol{Q}(h,\varphi,[0,\epsilon])$ for a small enough interval $[0,\epsilon] \subset I$.

Proof. Let $h \in C^{2,+}(\mathbb{S}^{n-1})$ and $\varphi \in C^2(\mathbb{S}^{n-1})$ be strictly positive even functions on $\mathbb{S}^{n-1}$; there exists $a > 0$ such that $h_s := h\varphi^s$ is the support function of a convex body Q_s for all $s \in [-a,a]$. Note that for $s,t \in [-a,a]$ we get

$$h_{\lambda s+(1-\lambda)t} = h_s^\lambda h_t^{1-\lambda},$$

and thus

$$Q_{\lambda s+(1-\lambda)t} = Q_s^\lambda Q_t^{1-\lambda}.$$

If the Conjecture 1.2 is true, then

$$\gamma(Q_{\lambda s+(1-\lambda)t}) = \gamma(Q_s^\lambda Q_t^{1-\lambda}) \geq \gamma(Q_s)^\lambda \gamma(Q_t)^{1-\lambda},$$

which means that $\gamma(Q_s)$ is log-concave in $[-a,a]$. □

4. Proof of Theorem 1.3

The following Lemma is the key step in proving Theorem 1.3. To prove it, we express a measure of a convex set in terms of its support function and run a long and technical computation, involving integration by parts; the complete proof is outlined in the Section 6.

Lemma 4.1. *Let $R > 0$. Let γ be a rotation invariant measure with density $f(|x|)$, and let $A = \int_0^1 t^{n-1} f(Rt)dt$. In the case $h_K = R$, (13) is equivalent to the validity of the following inequality for every $\psi \in C^2(\mathbb{S}^{n-1})$:*

$$\frac{Af(R)}{|\mathbb{S}^{n-1}|}\left((n-1)\int_{\mathbb{S}^{n-1}} \psi^2 du - \int_{\mathbb{S}^{n-1}} |\nabla_\sigma \psi|^2 du\right) + \frac{ARf'(R)}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \psi^2 du \leq \frac{n-1}{n} f(R)^2 \left(\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \psi du\right)^2. \tag{15}$$

By Lemma 3.1, to prove the Theorem, it suffices to show the validity of (15). Let us denote the quadratic operators appearing in the left-hand side and in the right-hand side of the inequality (15) by $B_1(\psi)$ and $B_2(\psi)$, correspondingly. That is,

$$B_1(\psi) = \frac{Af(R)}{|\mathbb{S}^{n-1}|}\left((n-1)\int_{\mathbb{S}^{n-1}} \psi^2 du - \int_{\mathbb{S}^{n-1}} |\nabla_\sigma \psi|^2 du\right) + \frac{ARf'(R)}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \psi^2 du,$$

and

$$B_2(\psi) = \frac{n-1}{n} f(R)^2 \left(\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \psi du\right)^2.$$

The next step is to decompose ψ as the sum of a constant function and a function which is orthogonal to constant functions. Let us write

$$\psi = \psi_0 + \psi_1$$

where

$$\psi_0 = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \psi du \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} \psi_1 du = 0.$$

Note that

$$\int_{\mathbb{S}^{n-1}} \psi^2 d\sigma = \int_{\mathbb{S}^{n-1}} \psi_0^2 d\sigma + \int_{\mathbb{S}^{n-1}} \psi_1^2 d\sigma.$$

Therefore,

$$B_1(\psi) = B_1(\psi_0) + B_1(\psi_1),$$

as well as

$$B_2(\psi) = B_2(\psi_0) + B_2(\psi_1).$$

Since γ is radially symmetric, one has $f' \leq 0$. Moreover, by the standard Poincaré inequality on the unit sphere,

$$(n-1)\int_{\mathbb{S}^{n-1}} \psi^2 du - \int_{\mathbb{S}^{n-1}} |\nabla_\sigma \psi|^2 du \leq 0, \tag{16}$$

for every ψ such that

$$\int_{\mathbb{S}^{n-1}} \psi du = 0. \tag{17}$$

Thus

$$B_1(\psi_1) \leq 0 = B_2(\psi_1).$$

To prove (15) it remains to show that

$$B_1(\psi_0) \leq B_2(\psi_0). \tag{18}$$

This condition is equivalent to

$$\gamma(\lambda r_1 B_2^n + (1-\lambda) r_2 B_2^n)^{\frac{1}{n}} \geq \lambda\gamma(r_1 B_2^n)^{\frac{1}{n}} + (1-\lambda)\gamma(r_2 B_2^n)^{\frac{1}{n}}, \tag{19}$$

for some $r_1, r_2 \in [R, R+\epsilon]$. As was shown in [27] (see also the third named author [28]), this statement follows from log-Brunn-Minkowski conjecture in the case of log-concave spherically invariant measures and when K and L are Euclidean balls. The latter is indeed true: it follows from the results of [15] and [36].

5. Proof of the Theorem 1.4

As before, we start with a Lemma, which shall be rigorously proved in Section 6.

Lemma 5.1. *Let $R > 0$. Let γ be a rotation invariant measure with density $f(|x|)$, and let $A = \int_0^1 t^{n-1} f(Rt)dt$. In the case $h_K = R$, (14) is equivalent to the following inequality:*

$$\begin{aligned} &A\,[nf(R) + Rf'(R)]\,\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \psi^2 du - Af(R)\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} |\nabla_\sigma \psi|^2 du \leq \\ &f(R)^2\left(\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \psi d\sigma\right)^2, \end{aligned} \tag{20}$$

for every even $\psi \in C^2(\mathbb{S}^{n-1})$.

We follow the argument of the previous section and split the proof into two cases.

Case 1. Consider an even $\psi \in C^2(\mathbb{S}^{n-1})$ such that $\int \psi = 0$. Here we use some basic facts from the theory of spherical harmonics, which can be found, for instance in [38, Appendix], where the reader will find hints to the corresponding literature. We denote by Δ_σ the spherical Laplace operator (or Laplace-Beltrami operator), on $\mathbb{S}^{n-1}$. The first eigenvalue of Δ_σ is 0, and the corresponding eigenspace if formed by constant functions. Hence the zero-mean condition on ψ implies that ψ is orthogonal to such eigenspace. The second eigenvalue of Δ_σ is $n-1$, and the corresponding eigenspace is formed by the restrictions of linear functions of $\mathbb{R}^n$ to $\mathbb{S}^{n-1}$. As each of them is odd and ψ is even, ψ is orthogonal to this eigenspace as well. Finally, the third eigenvalue is $2n$. Then the inequality (20) amounts to

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \psi^2 du \leq \frac{f(R)}{nf(R) + Rf'(R)}\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} |\nabla_\sigma \psi|^2 du. \tag{21}$$

Hence

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \psi^2 du \leq \frac{1}{2n}\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} |\nabla_\sigma \psi|^2 du. \tag{22}$$

Since f is decreasing, we have $f'(R) \leq 0$, and hence

$$\frac{f(R)}{nf(R) + Rf'(R)} \geq \frac{1}{n} > \frac{1}{2n}. \tag{23}$$

The inequalities (22) and (23) imply (21).

Case 2. Let ψ be a constant function.The inequality (20) holds for constant functions because, once again, the log-Brunn-Minkowski inequality holds in the case of spherically invariant measures and Euclidean balls.

To summarize, we established (20) separately for constant functions and centered functions. A polarization argument analogous to the one presented in the proof of Theorem 1.3 finishes the proof.

6. Auxiliary results

6.1. **A formula expressing a measure of a convex set in terms of its support function.** Let γ be a probability measure on $\mathbb{R}^n$; we assume that γ has a density F with respect to the Lebesgue measure, and that F is sufficiently regular (*e.g.* continuous). We leave the proof of the lemma below to the reader, as it is a standard argument involving polar coordinates.

Lemma 6.1. *Let K be a $C^{2,+}$ convex body; let h and H be the support function of K and its homogenous extension, respectively. Assume that the origin is in the interior of K. Then*

$$\gamma(K) = \int_{\mathbb{S}^{n-1}} h(y) \det Q(h;y) \int_0^1 t^{n-1} F\left(t\nabla H(y)\right) dt dy. \tag{24}$$

6.2. **The cofactor matrix and related notions.** Let $M = (m_{ij})$ be an $N \times N$ symmetric matrix, $N \in \mathbb{N}$. We define $C[M]$, the *cofactor matrix* of M, as follows

$$C[M] = (c_{ij}[M])_{i,j=1,\dots,N} \quad \text{where} \quad c_{ij}[M] = \frac{\partial \det}{\partial m_{ij}}(M) \quad i,j = 1,\dots,N.$$

$C[M]$ is an $N \times N$ symmetric matrix. Using the homogeneity of the determinant we get

$$\sum_{i,j=1}^{N} c_{ij}[M] m_{ij} = N \, \det(M). \tag{25}$$

We shall also consider the second derivatives of the determinant of a matrix with respect to its entries:

$$c_{ij,kl}[M] = \frac{\partial^2 \det}{\partial m_{ij} \partial m_{kl}}(M).$$

By homogeneity we have that, for every $i,j = 1,\dots,N$

$$\sum_{k,l=1}^{N} c_{ij,kl}[M] \, m_{kl} = (N-1) c_{ij}[M]. \tag{26}$$

6.3. **The Cheng-Yau lemma and an extension.** Let $h \in C^{2,+}(\mathbb{S}^{n-1})$, and assume additionally that $h \in C^3(\mathbb{S}^{n-1})$. Consider the cofactor matrix $y \to C[Q(h;y)]$. This is a matrix of functions on $\mathbb{S}^{n-1}$. The lemma of Cheng and Yau asserts that each column of this matrix is divergence-free.

Lemma 6.2 (Cheng-Yau.). *Let $h \in C^{2,+}(\mathbb{S}^{n-1}) \cap C^3(\mathbb{S}^{n-1})$. Then, for every index $j \in \{1,\dots,n-1\}$ and for every $y \in \mathbb{S}^{n-1}$,*

$$\sum_{i=1}^{n-1} \left(c_{ij}[Q(h;y)]\right)_i = 0,$$

where the sub-script i denotes the derivative with respect to the i-th element of an orthonormal frame on $\mathbb{S}^{n-1}$.

For simplicity of notation we shall often write $C(h)$, $c_{ij}(h)$ and $c_{ij,kl}(h)$ in place of $C[Q(h)]$, $c_{ij}[Q(h)]$ and $c_{ij,kl}[Q(h)]$ respectively.

As a corollary of the previous result we have the following integration by parts formula. If $h \in C^{2,+}(\mathbb{S}^{n-1}) \cap C^3(\mathbb{S}^{n-1})$ and $\psi, \phi \in C^2(\mathbb{S}^{n-1})$, then

$$\int_{\mathbb{S}^{n-1}} \phi \, c_{ij}(h)(\psi_{ij} + \psi \, \delta_{ij}) dy = \int_{\mathbb{S}^{n-1}} \psi \, c_{ij}(h)(\phi_{ij} + \phi \, \delta_{ij}) dy. \tag{27}$$

The Lemma of Cheng and Yau admits the following extension (see the paper by the first-named author, Hug and Saorin-Gomez [14]).

Lemma 6.3. *Let $\psi \in C^2(\mathbb{S}^{n-1})$ and $h \in C^{2,+}(\mathbb{S}^{n-1}) \cap C^3(\mathbb{S}^{n-1})$. Then, for every $k \in \{1,\dots,n-1\}$ and for every $y \in \mathbb{S}^{n-1}$*

$$\sum_{i=1}^{n-1} \left(c_{ij,kl}[Q(h;y)](\psi_{ij} + \psi\delta_{ij})\right)_l = 0.$$

Correspondingly we have, for every $h \in C^{2,+}(\mathbb{S}^{n-1}) \cap C^3(\mathbb{S}^{n-1})$, $\psi, \varphi, \phi \in C^2(\mathbb{S}^{n-1})$ and $i, j \in \{1,\dots,n-1\}$

$$\begin{aligned} &\int_{\mathbb{S}^{n-1}} \psi\, c_{ij,kl}(h)(\varphi_{ij} + \varphi\delta_{ij})((\phi)_{kl} + \phi\,\delta_{kl})dy \\ &= \int_{\mathbb{S}^{n-1}} \phi\, c_{ij,kl}(h)(\varphi_{ij} + \varphi\delta_{ij})((\psi)_{kl} + \psi\,\delta_{kl})dy. \end{aligned} \tag{28}$$

6.4. **Proof of the Lemma 4.1.** As usual, γ is a radially symmetric log-concave measure on $\mathbb{R}^n$, with density F with respect to Lebesgue measure; in particular, we write F in the form:

$$F(x) = f(|x|).$$

We will assume that f is smooth, more precisely $f \in C^2([0,\infty))$. Let us fix $h \in C^{2,+}(\mathbb{S}^{n-1})$ and let K be a convex body with support function h. Let $\psi \in C^2(\mathbb{S}^{n-1})$ and consider the one-parameter system of convex bodies $\mathbf{K}(h,\psi,[-a,a])$ for a suitably small $a > 0$. In particular for every $s \in [-a,a]$ there exists a convex body K_s such that $h_{K_s} = h_s$. Hence we may consider the function

$$g\,:\,[-a,a] \to \mathbb{R}, \quad g(s) = \gamma(K_s).$$

The aim of this subsection is to derive formulas for the first and second derivative of $g(s)$ at $s = 0$. We start from the expression:

$$g(s) = \int_{\mathbb{S}^{n-1}} h_s(u)\,\det(Q(h_s;u)) \int_0^1 t^{n-1} f(t\sqrt{h_s^2(u) + |\nabla_\sigma h_s(u)|^2})dtdu,$$

where we used Lemma 6.1, the rotation invariance of γ, and Remark 2.3. To simplify notations we set

$$\begin{aligned} &Q_s = Q(h_s;u)\,, \quad Q = Q_0; \quad D_s = \left[h_s^2(u) + |\nabla_\sigma h_s(u)|^2\right]^{1/2}, \quad D = D_0; \\ &A_s = \int_0^1 t^{n-1} f(tD_s)dt\,, \quad A = A_0; \quad B_s = \int_0^1 t^n f'(tD_s)dt\,, \quad B = B_0; \\ &C_s = \int_0^1 t^{n+1} f''(tD_s)dt\,, \quad C = C_0. \end{aligned}$$

Then

$$\begin{aligned} g'(s) \;&=\; \int_{\mathbb{S}^{n-1}} \psi \det(Q_s) A_s du + \int_{\mathbb{S}^{n-1}} h_s c_{ij}(h_s)(\psi_{ij} + \psi\delta_{ij}) A_s du \\ &+\; \int_{\mathbb{S}^{n-1}} h_s \det(Q_s) B_s \frac{h_s\psi + \langle \nabla_\sigma h_s, \nabla_\sigma \psi\rangle}{D_s} du. \end{aligned} \tag{29}$$

Passing to the second derivative (for $s=0$) we get

$$
\begin{aligned}
g''(0) &= 2\int_{\mathbb{S}^{n-1}} \psi c_{ij}(h)(\psi_{ij}+\psi\delta_{ij})A du \\
&+ 2\int_{\mathbb{S}^{n-1}} \psi \det(Q) B \frac{h\psi+\langle\nabla_\sigma h,\nabla_\sigma\psi\rangle}{D} du \\
&+ 2\int_{\mathbb{S}^{n-1}} h c_{ij}(h)(\psi_{ij}+\psi\delta_{ij}) B \frac{h\psi+\langle\nabla_\sigma h,\nabla_\sigma\psi\rangle}{D} du \\
&+ \int_{\mathbb{S}^{n-1}} A h c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})(\psi_{kl}+\psi\delta_{kl}) du \\
&+ \int_{\mathbb{S}^{n-1}} h \det(Q) C \left[\frac{h\psi+\langle\nabla_\sigma h,\nabla_\sigma\psi\rangle}{D}\right]^2 du \\
&+ \int_{\mathbb{S}^{n-1}} h \det(Q) B \left[D(h^2+|\nabla_\sigma\psi|^2) - \frac{[h\psi+\langle\nabla_\sigma h,\nabla_\sigma\psi\rangle]^2}{D}\right]\frac{1}{D^2} du. \qquad (30)
\end{aligned}
$$

We now focus on the fourth summand of the last expression. Applying formulas (28) and (26) we get

$$
\begin{aligned}
&\int_{\mathbb{S}^{n-1}} A h c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})(\psi_{kl}+\psi\delta_{kl}) du \\
=& \int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})((Ah)_{kl}+Ah\delta_{kl}) du \\
=& \int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})(A(h_{kl}+h\delta_{kl})+2A_k h_l + hA_{kl}) du \\
=& \int_{\mathbb{S}^{n-1}} A\psi c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})(h_{kl}+h\delta_{kl}) du \\
+& \int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})(2A_k h_l + hA_{kl}) du \\
=& (n-2)\int_{\mathbb{S}^{n-1}} A\psi c_{ij}(h)(\psi_{ij}+\psi\delta_{ij}) du \\
+& \int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})(2A_k h_l + hA_{kl}) du.
\end{aligned}
$$

Hence

$$
\begin{aligned}
g''(0) &= n\int_{\mathbb{S}^{n-1}} \psi c_{ij}(h)(\psi_{ij}+\psi\delta_{ij})A du + 2\int_{\mathbb{S}^{n-1}} \psi \det(Q) B \frac{h\psi+\langle\nabla_\sigma h,\nabla_\sigma\psi\rangle}{D} du \\
&+ 2\int_{\mathbb{S}^{n-1}} h c_{ij}(h)(\psi_{ij}+\psi\delta_{ij}) B \frac{h\psi+\langle\nabla_\sigma h,\nabla_\sigma\psi\rangle}{D} du \\
&+ \int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij}+\psi\delta_{ij})(2A_k h_l + hA_{kl}) du \\
&+ \int_{\mathbb{S}^{n-1}} h \det(Q) C \left[\frac{h\psi+\langle\nabla_\sigma h,\nabla_\sigma\psi\rangle}{D}\right]^2 du \\
&+ \int_{\mathbb{S}^{n-1}} h \det(Q) B \left[D(\psi^2+|\nabla_\sigma\psi|^2) - \frac{[h\psi+\langle\nabla_\sigma h,\nabla_\sigma\psi\rangle]^2}{D}\right]\frac{1}{D^2} du. \qquad (31)
\end{aligned}
$$

Let $h \equiv R$, $R > 0$. This choice considerably simplifies the situation as:

$$Q = RI_{n-1}; \quad \nabla_\sigma \equiv R; \quad D \equiv R; \quad c_{ij}(h) \equiv R^{n-2}\delta_{ij};$$
$$A = \int_0^1 t^{n-1} f(Rt)dt, \quad B = \int_0^1 t^n f'(Rt)dt, \quad C = \int_0^1 t^{n+1} f''(Rt)dt.$$

Here I_{n-1} denotes the $(n-1)\times(n-1)$ identity matrix. In particular A does not depend on the point u on $\mathbb{S}^{n-1}$, so that

$$A_i \equiv A_{ij} \equiv 0 \quad \text{on } \mathbb{S}^{n-1}.$$

Hence $g(0) = |\mathbb{S}^{n-1}|R^n A$, and

$$\begin{aligned} g'(0) &= R^{n-1}A\int_{\mathbb{S}^{n-1}} \psi du + R^{n-1}A\int_{\mathbb{S}^{n-1}} (\Delta_\sigma\psi + (n-1)\psi)du + R^n B\int_{\mathbb{S}^{n-1}} \psi du \\ &= R^{n-1}(nA + RB)\int_{\mathbb{S}^{n-1}} \psi du. \qquad (32) \end{aligned}$$

Here we used the fact that, by the divergence theorem on $\mathbb{S}^{n-1}$,

$$\int_{\mathbb{S}^{n-1}} \Delta_\sigma \psi du = 0.$$

As for the second derivative, we have

$$\begin{aligned} g''(0) &= nR^{n-2}A\int_{\mathbb{S}^{n-1}} \psi(\Delta_\sigma\psi + (n-1)\psi)du + 2R^{n-1}B\int_{\mathbb{S}^{n-1}} \psi^2 du \\ &+ 2R^{n-1}B\int_{\mathbb{S}^{n-1}} \psi(\Delta_\sigma\psi + (n-1)\psi))du + R^n C\int_{\mathbb{S}^{n-1}} \psi^2 du \\ &+ R^{n-1}B\int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du. \end{aligned}$$

By the divergence theorem,

$$\int_{\mathbb{S}^{n-1}} \psi\Delta_\sigma\psi du = -\int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du,$$

and thus

(33)
$$g''(0) = R^{n-2}(An(n-1) + 2nRB + R^2C)\int_{\mathbb{S}^{n-1}} \psi^2 du - R^{n-2}(nA + RB)\int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du.$$

Integrating by parts in t, we get

$$f(R) = nA + RB,$$

and

$$f'(R) = (n+1)B + RC.$$

Thus we obtain

$$g'(0) = R^{n-1}f(R)\int_{\mathbb{S}^{n-1}} \psi du, \qquad (34)$$

and

$$\begin{aligned} g''(0) &= R^{n-2}\left[(n-1)f(R) + Rf'(R)\right]\int_{\mathbb{S}^{n-1}} \psi^2 du - R^{n-2}f(R)\int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du \\ &= R^{n-2}f(R)\left((n-1)\int_{\mathbb{S}^{n-1}} \psi^2 du - \int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du\right) + R^{n-1}f'(R)\int_{\mathbb{S}^{n-1}} \psi^2 du. \qquad (35) \end{aligned}$$

This concludes the proof of Lemma 4.1.

6.5. **Proof of the Lemma 5.1.** Firstly, we state the following.

Lemma 6.4. *Let $n \geq 2$. Let γ be a measure on $\mathbb{R}^n$. Fix $h \in C^{2,+}(\mathbb{S}^{n-1})$, $\varphi \in C^2(\mathbb{S}^{n-1})$, $\varphi > 0$ and set $\psi = h \log \varphi$. Let $\boldsymbol{K}(h, \psi, I)$, with $I = [-a, a]$ and $a > 0$, be the corresponding one-parameter family. Consider the function $f(s) = \gamma(K_s)$. Introduce the additional notation for the operator $F(h, \psi) := f'(0)$. Set*

$$A(h,\psi) := \left.\frac{dF\left(h, \frac{h+s\psi}{h}\psi\right)}{ds}\right|_{s=0}. \tag{36}$$

Consider the one-parameter family $\boldsymbol{Q}(h, \varphi, [-a, a])$, i.e. the collection of sets with support functions $h_s = h\varphi^s$, $s \in [-a, a]$. Let $g(s) = \gamma(Q_s)$. Then

- *$g(0) = f(0)$;*
- *$g'(0) = f'(0)$;*
- *$g''(0) = f''(0) + A(h, \psi)$.*

The proof of the Lemma immediately follows from the fact that

$$h\varphi^s = h + sh\log\varphi + o(s), \quad \text{as } s \to 0,$$

with the selection $\psi = h \log \varphi$. When $h \equiv R > 0$, the additional term introduced in Lemma 6.4 can be written as follows:

$$A(h,\psi) = f(R)\int_{\mathbb{S}^{n-1}} \psi^2 du.$$

That, together with Lemma 4.1, implies Lemma 5.1.

6.6. **Additional remarks.** Finally, we note that Lemma 6.4 implies the following result.

Theorem 6.5 (Infinitesimal form of Log-Brunn-Minkowski conjecture)**.** *Let $n \geq 2$ be an integer. If Conjecture 1.2 is true, then for every $h \in C_e^{2,+}(\mathbb{S}^{n-1})$, $\psi \in C^2(\mathbb{S}^{n-1})$, ψ even and strictly positive,*

$$\int_{\mathbb{S}^{n-1}} \psi^2 \frac{1+\operatorname{tr}(Q^{-1}(h))h}{h^2} d\bar{V}_h - n\left(\int_{\mathbb{S}^{n-1}} \frac{\psi}{h} d\bar{V}_h\right)^2 \leq \int_{\mathbb{S}^{n-1}} \frac{1}{h}\langle Q^{-1}(h)\nabla\psi, \nabla\psi\rangle d\bar{V}_h. \tag{37}$$

Here h is the support function of K, $Q(h)$ is the curvature matrix of K and

$$d\bar{V}_h = \frac{1}{|K|}\frac{1}{n} h_K(u)\det Q(h_K(u))du$$

is the normalized cone measure of the convex body K.

A corresponding infinitesimal Brunn-Minkowski inequality for Lebesgue measure was obtained by the first named author in [11] and reads as:

$$\int_{\mathbb{S}^{n-1}} \psi^2 \frac{\operatorname{tr}(Q^{-1}(h))}{h} d\bar{V}_h - (n-1)\left(\int_{\mathbb{S}^{n-1}} \frac{\psi}{h} d\bar{V}_h\right)^2 \leq \int_{\mathbb{S}^{n-1}} \frac{1}{h}\langle Q^{-1}(h)\nabla\psi, \nabla\psi\rangle d\bar{V}_h. \tag{38}$$

Note that by the Cauchy-Schwarz inequality,

$$\int_{\mathbb{S}^{n-1}} \frac{\psi^2}{h^2} d\bar{V}_h \geq \left(\int_{\mathbb{S}^{n-1}} \frac{\psi}{h} d\bar{V}_h\right)^2.$$

Hence, (37) is indeed a strengthening of (38).

In particular, letting $\varphi \equiv 1$ we arrive to the following corollary of Theorem 6.5.

Corollary 6.6 (A strengthening of Minkowski's second inequality.)**.** *Let K be a convex symmetric set in the plane, or a convex unconditional set in $\mathbb{R}^n$. Then,*

$$V_n(K)\left(V_{n-2}(K)+\int_{\partial K}\frac{1}{\langle y,\nu_K(y)\rangle}d\sigma(y)\right)\leq V_{n-1}(K)^2, \tag{39}$$

where V_{n-i} are the intrinsic volumes of K, $\nu_K(y)$ stands for the unit normal at $y\in\partial K$ and $d\sigma(y)$ is the surface area measure on ∂K.

Minkowski's second inequality, which states that for every convex set $K\subset\mathbb{R}^n$ one has

$$V_n(K)V_{n-2}(K)\leq\frac{n-1}{n}V_{n-1}(K)^2,$$

is deduced from (39) by using the Cauchy-Schwarz inequality. For a more general version of this inequality see, for example, Schneider [38, Chapter 4].

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