Do Minkowski averages get progressively more convex?

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Abstract

Let us define, for a compact set $A \subset \mathbf{R}^n$, the Minkowski averages of A:

$$A(k) = \left\{\frac{a_1 + \dots + a_k}{k} : a_1, \dots, a_k \in A\right\} = \frac{1}{k} \left(\underbrace{A + \dots + A}_{k \text{ times}}\right).$$

We study the monotonicity of the convergence of A(k) towards the convex hull of A, when considering the Hausdorff distance, the volume deficit and a non-convexity index of Schneider as measures of convergence. For the volume deficit, we show that monotonicity fails in general, thus disproving a conjecture of Bobkov, Madiman and Wang. For Schneider's non-convexity index, we prove that a strong form of monotonicity holds, and for the Hausdorff distance, we establish that the sequence is eventually nonincreasing.

1 Introduction

This note announces and proves some of the results obtained in [3]. Let us denote for a compact set $A \subset \mathbf{R}^n$ and for a positive integer k,

$$A(k) = \left\{\frac{a_1 + \dots + a_k}{k} : a_1, \dots, a_k \in A\right\} = \frac{1}{k} \left(\underbrace{A + \dots + A}_{k \text{ times}}\right).$$
(1)

Denoting by conv(A) the convex hull of A, and by

 $d(A) := \inf\{r > 0 : \operatorname{conv}(A) \subset A + rB_2^n\}$

the Hausdorff distance between a set A and its convex hull, it is a classical fact (proved independently by [7, 2] in 1969, and often called the Shapley-Folkmann-Starr theorem) that A(k)converges in Hausdorff distance to conv(A) as $k \to \infty$. Furthermore [7, 2] also determined

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the rate of convergence: it turns out that d(A(k)) = O(1/k) for any compact set A. For sets of nonempty interior, this convergence of Minkowski averages to the convex hull can also be expressed in terms of the volume deficit $\Delta(A)$ of a compact set A in \mathbb{R}^n , which is defined as:

$$\Delta(A) := \operatorname{Vol}_n(\operatorname{conv}(A) \setminus A) = \operatorname{Vol}_n(\operatorname{conv}(A)) - \operatorname{Vol}_n(A),$$

where Vol_n denotes Lebesgue measure in \mathbb{R}^n . It was shown by [2] that if A is compact with nonempty interior, then the volume deficit of A(k) also converges to 0; more precisely, $\Delta(A(k)) = O(1/k)$ for any compact set A with nonempty interior.

Our original motivation came from a conjecture made by Bobkov, Madiman and Wang [1]:

Conjecture 1.1 ([1]). Let A be a compact set in \mathbb{R}^n for some $n \in \mathbb{N}$, and let A(k) be defined as in (1). Then the sequence $\Delta(A(k))$ is non-increasing in k, or equivalently, $\{\operatorname{Vol}_n(A(k))\}_{k\geq 1}$ is non-decreasing.

We show that Conjecture 1.1 fails to hold in general, even for moderately high dimension.

Theorem 1.1. Conjecture 1.1 is false in \mathbb{R}^n for $n \ge 12$, and true for \mathbb{R}^1 .

Notice that Conjecture 1.1 remains open for 1 < n < 12. In particular, the arguments presented in this note do not seem to work. In analogy with Conjecture 1.1, we also consider whether one can have monotonicity of $\{c(A(k))\}_{k\geq 1}$, where c is a non-convexity index defined by Schneider [6] as follows:

$$c(A) := \inf\{\lambda \ge 0 : A + \lambda \operatorname{conv}(A) \text{ is convex}\}\$$

A nice property of Schneider's index is that it is affine-invariant, i.e., c(TA + x) = c(A) for any nonsingular linear map T on \mathbb{R}^n and any $x \in \mathbb{R}^n$.

Contrary to the volume deficit, we prove that Schneider's non-convexity index c satisfies a strong kind of monotonicity in any dimension.

Theorem 1.2. Let A be a compact set in \mathbb{R}^n and $k \in \mathbb{N}^*$. Then

$$c(A(k+1)) \le \frac{k}{k+1}c(A(k)).$$

Finally, we also prove that eventually, for $k \ge c(A)$, the Hausdorff distance between A(k) and conv(A) is also strongly decreasing.

Theorem 1.3. Let A be a compact set in \mathbb{R}^n and $k \ge c(A)$ be an integer. Then

$$d\left(A(k+1)\right) \le \frac{k}{k+1}d\left(A(k)\right).$$

Moreover, Schneider proved in [6] that $c(A) \leq n$ for every compact subset A of \mathbb{R}^n . It follows that the eventual monotonicity of the sequence d(A(k)) holds true for $k \geq n$.

It is natural to ask what the relationship is in general between convergence of c, Δ and d to 0, for arbitrary sequences (C_k) of compact sets. In fact, none of these 3 notions of approach

to convexity are comparable with each other in general. To see why, observe that while c is scaling-invariant, neither Δ nor d are; so it is easy to construct examples of sequences (C_k) such that $c(C_k) \to 0$ but $\Delta(C_k)$ and $d(C_k)$ remain bounded away from 0. The same argument enables us to construct examples of sequences (C_k) such that $c(C_k)$ remain bounded away from 0, whereas $\Delta(C_k)$ and $d(C_k)$ converge to 0. Furthermore, $\Delta(C_k)$ remains bounded away from 0 for any sequence C_k of finite sets, whereas $c(C_k)$ and $d(C_k)$ could converge to 0 if the finite sets form a finer and finer grid filling out a convex set. An example where $\Delta(C_k) \to 0$ but both $c(C_k)$ and $d(C_k)$ are bounded away from 0 is given by taking a 3-point set with 2 of the points getting arbitrarily closer but staying away from the third. One can obtain further relationships between these measures of non-convexity if further conditions are imposed on the sequence C_k ; details may be found in [3].

The rest of this note is devoted to the examination of whether A(k) becomes progressively more convex as k increases, when measured through the functionals Δ , d and c. The concluding section contains some additional discussion.

2 The behavior of volume deficit

We prove Theorem 1.1 in this section. We start by constructing a counterexample to the conjecture in \mathbb{R}^n , for $n \geq 12$. Let F be a p-dimensional subspace of \mathbb{R}^n , where $p \in \{1, \ldots, n-1\}$. Let us consider $A = I_1 \cup I_2$, where $I_1 \subset F$ and $I_2 \subset F^{\perp}$ are convex sets, and F^{\perp} denotes the orthogonal complement of F. One has

$$A + A = 2I_1 \cup (I_1 \times I_2) \cup 2I_2,$$
$$A + A + A = 3I_1 \cup (2I_1 \times I_2) \cup (I_1 \times 2I_2) \cup 3I_2.$$

Notice that

$$\operatorname{Vol}_n(A+A) = \operatorname{Vol}_p(I_1)\operatorname{Vol}_{n-p}(I_2),$$
$$\operatorname{Vol}_n(A+A+A) = \operatorname{Vol}_p(I_1)\operatorname{Vol}_{n-p}(I_2)(2^p + 2^{n-p} - 1).$$

Thus, $\operatorname{Vol}_n(A(3)) \ge \operatorname{Vol}_n(A(2))$ if and only if

$$2^{p} + 2^{n-p} - 1 \ge \left(\frac{3}{2}\right)^{n}.$$
(2)

Notice that inequality (2) does not hold when $n \ge 12$ and $p = \lceil \frac{n}{2} \rceil$.

For \mathbf{R}^1 , the conjecture may be proved by adapting a proof of [4] on cardinality of integer sumsets; this was also independently observed by F. Barthe. Let $k \ge 1$. Set $S = A_1 + \cdots + A_k$ and for $i \in [k]$, let $a_i = \min A_i$, $b_i = \max A_i$,

$$S_i = \sum_{j \in [k] \setminus \{i\}} A_j$$

 $s_i = \sum_{j < i} a_j + \sum_{j > i} b_j$, $S_i^- = \{x \in S_i; x \le s_i\}$ and $S_i^+ = \{x \in S_i; x > s_i\}$. For all $i \in [k-1]$, one has

$$S \supset (a_i + S_i^-) \cup (b_{i+1} + S_{i+1}^+).$$

Since $a_i + s_i = \sum_{j \le i} a_j + \sum_{j > i} b_j = b_{i+1} + s_{i+1}$, the above union is a disjoint union. Thus for $i \in [k-1]$

$$\operatorname{Vol}_1(S) \ge \operatorname{Vol}_1(a_i + S_i^-) + \operatorname{Vol}_1(b_{i+1} + S_{i+1}^+) = \operatorname{Vol}_1(S_i^-) + \operatorname{Vol}_1(S_{i+1}^+).$$



Figure 1: A counterexample in \mathbf{R}^{12} .

Notice that $S_1^- = S_1$ and $S_k^+ = S_k \setminus \{s_k\}$, thus adding the above k-1 inequalities we obtain

$$(k-1)\operatorname{Vol}_{1}(S) \geq \sum_{i=1}^{k-1} \left(\operatorname{Vol}_{1}(S_{i}^{-}) + \operatorname{Vol}_{1}(S_{i+1}^{+}) \right) = \operatorname{Vol}_{1}(S_{1}^{-}) + \operatorname{Vol}_{1}(S_{k}^{+}) + \sum_{i=2}^{k-1} \operatorname{Vol}_{1}(S_{i})$$
$$= \sum_{i=1}^{k} \operatorname{Vol}_{1}(S_{i}).$$

Now taking all the sets $A_i = A$, and dividing through by k(k-1), we see that we have established Conjecture 1.1 in dimension 1.

3 The behavior of Schneider's non-convexity index and the Hausdorff distance

We establish Theorems 1.2 and 1.3 in this section. This relies crucially on the elementary observations that $\operatorname{conv}(A+B) = \operatorname{conv}(A) + \operatorname{conv}(B)$ and $(t+s)\operatorname{conv}(A) = t\operatorname{conv}(A) + s\operatorname{conv}(A)$ for any t, s > 0 and any compact sets A, B.

Proof of Theorem 1.2. Denote $\lambda = c(A(k))$. Since $\operatorname{conv}(A(k)) = \operatorname{conv}(A)$, from the definition of c, one knows that $A(k) + \lambda \operatorname{conv}(A) = \operatorname{conv}(A) + \lambda \operatorname{conv}(A) = (1 + \lambda) \operatorname{conv}(A)$. Using that $A(k+1) = \frac{A}{k+1} + \frac{k}{k+1}A(k)$, one has

$$\begin{aligned} A(k+1) + \frac{k}{k+1}\lambda \operatorname{conv}(A) &= \frac{A}{k+1} + \frac{k}{k+1}A(k) + \frac{k}{k+1}\lambda \operatorname{conv}(A) \\ &= \frac{A}{k+1} + \frac{k}{k+1}\operatorname{conv}(A) + \frac{k}{k+1}\lambda \operatorname{conv}(A) \\ &\supset \frac{\operatorname{conv}(A)}{k+1} + \frac{k}{k+1}A(k) + \frac{k}{k+1}\lambda \operatorname{conv}(A) \\ &= \frac{\operatorname{conv}(A)}{k+1} + \frac{k}{k+1}(1+\lambda)\operatorname{conv}(A) \\ &= \left(1 + \frac{k}{k+1}\lambda\right)\operatorname{conv}(A). \end{aligned}$$

Since the other inclusion is trivial, we deduce that $A(k+1) + \frac{k}{k+1}\lambda \operatorname{conv}(A)$ is convex which proves that

$$c(A(k+1)) \le \frac{k}{k+1}\lambda = \frac{k}{k+1}c(A(k)).$$

Proof of Theorem 1.3. Let $k \ge c(A)$, then, from the definitions of c(A) and d(A(k)), one has

$$\operatorname{conv}(A) = \frac{A}{k+1} + \frac{k}{k+1}\operatorname{conv}(A) \quad \subset \quad \frac{A}{k+1} + \frac{k}{k+1}\left(A(k) + d(A(k))B_2^n\right) \\ = \quad A(k+1) + \frac{k}{k+1}d(A(k))B_2^n.$$

We conclude that

$$d\left(A(k+1)\right) \leq \frac{k}{k+1}d\left(A(k)\right).$$

4 Discussion

- 1. By repeated application of Theorem 1.2, it is clear that the convergence of c(A(k)) is at a rate O(1/k) for any compact set $A \subset \mathbf{R}^n$; this observation appears to be new. In [3], we study the question of the monotonicity of A(k), as well as convergence rates, when considering several different ways to measure non-convexity, including some not mentioned in this note.
- 2. Some of the results in this note are of interest when one is considering Minkowski sums of different compact sets, not just sums of A with copies of itself. Indeed, the original conjecture of [1] was of this form, and would have provided a strengthening of the classical Brunn-Minkowski inequality for more than 2 sets; of course, that conjecture is false since the weaker Conjecture 1.1 is false. Nonetheless we do have some related observations in [3]; for instance, it turns out that in general dimension, for compact sets A_1, \ldots, A_k ,

$$\operatorname{Vol}_n\left(\sum_{i=1}^k A_i\right) \ge \frac{1}{k-1} \sum_{i=1}^k \operatorname{Vol}_n\left(\sum_{j \in [k] \setminus \{i\}} A_j\right).$$

For convex sets B_i , an even stronger fact is true (that this is stronger may not be immediately obvious, but if follows from well known results, see, e.g., [5]):

$$\operatorname{Vol}_n(B_1 + B_2 + B_3) + \operatorname{Vol}_n(B_1) \ge \operatorname{Vol}_n(B_1 + B_2) + \operatorname{Vol}_n(B_1 + B_3).$$

3. There is a variant of the strong monotonicity of Schneider's index when dealing with different sets. If A, B, C are subsets of \mathbb{R}^n , then it is shown in [3] (by a similar argument to that used for Theorem 3) that $c(A + B + C) \leq \max\{c(A + B), c(B + C)\}$.

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