Homework #1 - Random Variables

Examples of random variables: Recall that a random variable X

1. has a Bernoulli distribution with parameter $p \in [0,1]$ if X takes values in $\{0,1\}$ and
$$\mathbb{P}(X=1)=p, \quad \mathbb{P}(X=0)=1-p.$$
2. has a binomial distribution with parameters $n \geq 1$ and $p \in [0,1]$ if X takes values in $\{0,1,\ldots,n\}$ and
$$\mathbb{P}(X=k)=\binom{n}{k}p^k(1-p)^{n-k}, \quad k \in \{0,\ldots,n\}.$$
3. has a geometric distribution with parameter $p>0$ if X takes values in $\mathbb{N} \setminus \{0\}$ and
$$\mathbb{P}(X=k)=p(1-p)^{k-1}, \quad k \in \mathbb{N} \setminus \{0\}.$$
4. has a Poisson distribution with parameter $\lambda > 0$ if X takes values in $\mathbb{N}$ and
$$\mathbb{P}(X=k)=e^{-\lambda}\frac{\lambda^k}{k!}, \quad k \in \mathbb{N}.$$
5. has an exponential distribution with parameter $\lambda > 0$ if X has the probability density function
$$f(x)=\lambda e^{-\lambda x}1_{[0,+\infty)}(x).$$
6. has a uniform distribution on a segment $[a,b]$ if X has the probability density function
$$f(x)=\frac{1}{b-a}1_{[a,b]}(x).$$
7. has a Gaussian distribution with parameters $m \in \mathbb{R}$ and $\sigma > 0$ if X has the probability density function
$$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-m)^2}{2\sigma^2}}.$$
8. has a Cauchy distribution with parameters $\alpha \in \mathbb{R}$ and $\beta > 0$ if X has the probability density function
$$f(x)=\frac{1}{\pi\beta}\frac{1}{1+(\frac{x-\alpha}{\beta})^2}.$$

Exercise 1. Compute the expectation and variance of the following discrete distributions: Bernoulli, binomial, geometric, Poisson.

Exercise 2. Compute the expectation and variance of the following continuous distributions: exponential, uniform on a segment $[a,b]$, Gaussian, Cauchy.

Exercise 3. A store owner possesses a stock of s items at time t_0. The demand during the period of time $[t_0, t_1]$ is expressed as a discrete random variable X with values in $\mathbb{N}$ having the following distribution

$$\mathbb{P}(X = k) = Cp(1-p)^k, \quad k \geq k_0,$$

where $p \in (0,1)$ and k_0 is a natural number in $\{0, \dots, s\}$.

1. Compute C and $\mathbb{E}[X]$.
2. If X is less than the stock s, then the remaining items are sold at a loss, and at a cost of c_1 dollars to the owner. If X is greater than s, then the owner has to order more items, at a cost of c_2 dollars to the owner.

   Compute the expectation of extra expenses that will occur to the owner during the period of time $[t_0, t_1]$.

Exercise 4. Consider a sequence $\{X_n\}$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote, for all $\varepsilon > 0$ and $k \geq 1$,

$$B_{\varepsilon,k} = \{\omega \in \Omega : |X_k(\omega)| < \varepsilon\}.$$

1. Fix $\varepsilon > 0$. Write with sets operations (intersection, union) on $B_{\varepsilon,k}$ the following sets:

$$A_{\varepsilon,n} = \{\omega \in \Omega : \forall k \geq n, |X_k(\omega)| < \varepsilon\}.$$

$$A_{\varepsilon} = \{\omega \in \Omega : \exists n \in \mathbb{N}, \forall k \geq n, |X_k(\omega)| < \varepsilon\}.$$

2. Show that the set

$$A = \{\omega \in \Omega : \lim_{k \to +\infty} X_k(w) = 0\}$$

   can be written with sets operations on $B_{\varepsilon,k}$.

Exercise 5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let A_n be a sequence such that for all $n \geq 1$, $\mathbb{P}(A_n) = \frac{1}{n^2}$. Find the probability that A_n occurs infinitely many often, that is find $\mathbb{P}(\cap_{n \geq 1} \cup_{k \geq n} A_k)$.

Exercise 6. Prove that for all random variable X, one has

$$\mathbb{E}[|X|] = \int_0^{+\infty} \mathbb{P}(|X| \geq t)\, dt.$$