Homework #1 – **Review of Expectation and Conditional Expectation**

Exercise 1.

Let $N\colon (\Omega, \mathcal{F}) \to (\mathbb{N}, \mathcal{P}(\mathbb{N}))$ be a random variable with values in $\mathbb{N}$, and let $\{X_n\}_{n\geq 1}\colon (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a sequence of random variables.

Show that X_N and $\sum_{k=1}^{N} X_k$ are random variables.

Exercise 2.

Let $X, Y\colon (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables such that $X \leq Y$ a.s.

1. Show that $\mathbb{E}[X] \leq \mathbb{E}[Y]$.
2. Show that if $\mathbb{E}[X] = \mathbb{E}[Y]$, then $X = Y$ a.s.

Exercise 3.

Let $X\colon (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable. Show that

$$\forall A \in \mathcal{F},\ \mathbb{E}[X 1_A] = 0 \quad \Longrightarrow \quad X = 0 \text{ a.s.}$$

Exercise 4.

Let $\{X_n\}_{n\geq 1}$ be a decreasing sequence of non-negative random variables such that $X_1 \in L^1$ (that is $\mathbb{E}[|X_1|] < +\infty$). Show that $\{X_n\}$ converges to X a.s. for some random variable X, and

$$\lim_{n\to+\infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

Is the statement still valid if $\mathbb{E}[|X_1|] = +\infty$?

Exercise 5. (Conditional expectation in L^1)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{G}$ be a sub-σ-algebra. Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. The conditional expectation of X with respect to $\mathcal{G}$, denoted by $\mathbb{E}[X|\mathcal{G}]$, is a random variable Y such that

1. Y is $\mathcal{G}$-measurable.
2. $\mathbb{E}[X 1_A] = \mathbb{E}[Y 1_A]$, for all $A \in \mathcal{G}$.

Prove all the properties of the conditional expectation when $X \in L^1$ (linearity, monotonicity, conditional Jensen's inequality, tower property, ...).

Exercise 6.

Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

1. Determine $\mathbb{E}[X|\{\emptyset, \Omega\}]$.
2. Determine $\mathbb{E}[X|\mathcal{P}(\Omega)]$.
3. Determine $\mathbb{E}[X|\mathcal{F}]$.
4. Let $A \in \mathcal{F}$. Determine $\mathbb{E}[X|\sigma(A)]$.

Exercise 7.

Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \text{Unif}([0,1]))$ and let $\mathcal{G} = \sigma([0, \frac{1}{2}])$. For $X \in L^1(\mathcal{B}([0,1]))$, determine $\mathbb{E}[X|\mathcal{G}]$.

Exercise 8. (Beppo-Levi's monotone convergence)
Let $\{X_n\}_{n\geq 1}$ be a non-decreasing sequence of non-negative random variables such that $\lim_n X_n = X$. Then,

$$\lim_{n\to+\infty} \mathbb{E}[X_n|\mathcal{G}] = \mathbb{E}[\lim_{n\to+\infty} X_n|\mathcal{G}] \quad \text{a.s.}$$

Exercise 9. (Fatou's lemma)
Let $\{X_n\}_{n\geq 1}$ be a sequence of non-negative random variables. Then,

$$\mathbb{E}[\liminf_{n\to+\infty} X_n|\mathcal{G}] \leq \liminf_{n\to+\infty} \mathbb{E}[X_n|\mathcal{G}] \quad \text{a.s.}$$

<u>**Hint:**</u> One may use Beppo-Levi's monotone convergence.

Exercise 10. (Lebesgue dominated convergence)
Let $\{X_n\}_{n\geq 1}$ be a sequence of integrable random variables. Assume that $\{X_n\}$ convergence to X a.s., and that there exists Y integrable such that $\forall n, |X_n| \leq Y$ a.s. Then,

$$\lim_{n\to+\infty} \mathbb{E}[X_n|\mathcal{G}] = \mathbb{E}[X|\mathcal{G}] \quad \text{a.s.}$$

<u>**Hint:**</u> One may note that $X_n + Y \geq 0, Y - X_n \geq 0$, and use Fatou's lemma.

Exercise 11.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $A \in \mathcal{F}$, and let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Find $\sigma(1_A)$, and Determine $\mathbb{E}[X|1_A]$.

Exercise 12.

Let X_1, X_2 be independent random variables having Poisson distribution with parameter λ_1, λ_2 respectively.

Find the distribution of the random variable $\mathbb{E}[X_1|X_1 + X_2]$.

Exercise 13.

Let X be a Gaussian random variable $\mathcal{N}(\mu, \sigma^2)$. Let Y be a standard Gaussian random variable $\mathcal{N}(0,1)$ independent of X.

Determine $\mathbb{E}[X + Y|X]$ and $\mathbb{E}[X|X + Y]$.