Homework #1 — **Stochastic Differential Equations (SDE) and Girsanov's theorem**

Exercise 1.
Show that the exponential SDE

$$dX_t = A_t X_t dB_t, \quad X_0 = x_0$$

has the following solution

$$X_t = x_0 e^{-\frac{1}{2}\int_0^t A_s^2 ds + \int_0^t A_s dB_s}.$$

Hint: Apply Ito's formula to $f(Y_t)$ where $f(x) = x_0 e^x$ and $Y_t = -\frac{1}{2}\int_0^t A_s^2 ds + \int_0^t A_s dB_s$.

Exercise 2. Among the following random variables, state for each pair whether they are absolutely continuous to each other or not:
X_1 Gausian with mean $m = 1$ and variance $\sigma^2 = 2$;
X_2 Binomial with parameter $n = 3$, $p = \frac{1}{2}$;
X_3 Poisson distribution with parameter $\lambda = 1$;
X_4 uniform on $[0, 1]$.

Exercise 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{B_t\}$ be a standard Brownian motion with $B_0 = 1$ under the measure $\mathbb{P}$. Define $\tau = \inf\{t > 0 : B_t = 0\}$, and $M_t = B_{t\wedge\tau}$.

1. Show that $\{M_t\}$ is a nonnegative martingale, solution of the SDE

$$dM_t = A_t M_t dB_t \qquad t < \tau,$$

   where $A_t = \frac{1}{M_t}$.

2. Consider $\mathbb{Q}$ the change of measure of $\mathbb{P}$ with respect to M_t. Show that $\{B_t\}$ is a Bessel process, that is, it is solution of the Bessel SDE

$$dX_t = \frac{1}{X_t} dt + dW_t,$$

   where $\{W_t\}$ is a standard Brownian motion under the new measure $\mathbb{Q}$.

Exercise 4. Suppose $\{B_t\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. For each of the following choices of X_t, find an equivalent probability measure $\mathbb{Q}$ such that the X_t is a standard Brownian motion in the new measure. Assume $X_0 = B_0 = 0$.

$$dX_t = 2dt + dB_t,$$

$$dX_t = 2dt + 6dB_t,$$

Exercise 5. Let $f\colon [0,+\infty)\times\mathbb{R}\to\mathbb{R}$ be a function of class $C^{1,2}([0,+\infty)\times\mathbb{R})$. Assume that f is solution of the heat equation

$$\frac{\partial}{\partial t}f(t,x) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}f(t,x).$$

Let $\{B_t\}$ be a standard Brownian motion.

1. Find the SDE solved by $f(t,B_t)$.
2. Deduce that $f(t,B_t)$ is a local martingale if and only if f is solution of the heat equation.
3. Show that if $\frac{\partial}{\partial x}f(t,x)$ is bounded, then $f(t,B_t)$ is a martingale.