Homework #1

Exercise 1. Consider a sequence $\{X_n\}$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote, for all $\varepsilon > 0$ and $k \geq 1$,

$$B_{\varepsilon,k} = \{\omega \in \Omega : |X_k(\omega)| < \varepsilon\}.$$

1. Fix $\varepsilon > 0$. Write with sets operations (intersection, union) on $B_{\varepsilon,k}$ the following sets:
$$A_{\varepsilon,k} = \{\omega \in \Omega : \forall k \geq n, |X_k(\omega)| < \varepsilon\}.$$
$$A_\varepsilon = \{\omega \in \Omega : \exists n \in \mathbb{N}, \forall k \geq n, |X_k(\omega)| < \varepsilon\}.$$
2. Show that the set
$$A = \{\omega \in \Omega : \lim_{k \to +\infty} X_k(w) = 0\}$$
can be written with sets operations on $B_{\varepsilon,k}$.

Exercise 2. Let $\{A_n\}$ be a sequence of subsets of Ω. Find $\limsup A_n$ and $\liminf A_n$ in the following cases:

1. for all $p \in \mathbb{N}$, $A_{2p} = F$ and $A_{2p+1} = G$, where $F, G \subset \Omega$.
2. for all $p \geq 1$, $A_{2p} = (-\infty, 1 + \frac{1}{2p})$ and $A_{2p+1} = (-\infty, -1 - \frac{1}{2p+1})$.

Exercise 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let A_n be a sequence such that for all $n \geq 1$, $\mathbb{P}(A_n) = \frac{1}{n^2}$. Find the probability that A_n occurs infinitely many often.

Exercise 4. Let $\{A_n\}$ be a sequence of events, all of probability 1. Show that $\cap_n A_n$ has probability 1.

Exercise 5. Let X be a non-negative random variable. Define, for $n \geq 1$ and $\omega \in \Omega$,

$$X_n(\omega) = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} 1_{[\frac{k-1}{2^n}, \frac{k}{2^n})}(X(\omega)) + n 1_{[n,+\infty)}(X(\omega)).$$

Show that $\{X_n\}$ is an increasing sequence of non-negative random variables, such that $\{X_n\}$ converges to X pointwise.

Exercise 6. Prove the **Jensen inequality:**

For all convex function ϕ and random variable X,

$$\mathbb{E}[\phi(X)] \geq \phi(\mathbb{E}[X]),$$

provided $\mathbb{E}[\phi(X)]$ and $\phi(\mathbb{E}[X])$ exist.

Exercise 7. A store owner possesses a stock of s items at time t_0. The demand during the period of time $[t_0, t_1]$ is expressed as a discrete random variable X with values in $\mathbb{N}$ having the following distribution

$$\mathbb{P}(X = k) = Cp(1-p)^k, \quad k \geq k_0,$$

where $p \in (0, 1)$ and k_0 is a natural number in $\{0, \dots, s\}$.

1. Compute C and $\mathbb{E}[X]$.

2. If X is less than the stock s, then the remaining items are sold at a loss, and at a cost of c_1 dollars to the owner. If X is greater than s, then the owner has to order more items, at a cost of c_2 dollars to the owner.

   Compute the expectation of extra expenses that will occur to the owner during the period of time $[t_0, t_1]$.

Exercise 8. Prove Fatou's lemma and Lebesgue dominated convergence theorem.
Hint: You can use the monotone convergence theorem.