Homework #2 — **Isotropic log-concave distributions**

Exercise 1. log-concave functions and log-concave random vectors

We say that $f\colon \mathbb{R}^n \to [0,+\infty)$ is **log-concave** if $\log(f)\colon \mathbb{R}^n \to [-\infty,+\infty)$ is a concave function. Recall that a random vector X in $\mathbb{R}^n$ is **log-concave** if it has a density probability function f_X with respect to Lebesgue measure in $\mathbb{R}^n$ such that f_X is log-concave.

- Some equivalent definitions:

  1. Show that f is log-concave if and only if for every $x, y \in \mathbb{R}^n$,
  $$f((1-\lambda)x+\lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}.$$
  2. Show that f is log-concave if and only if $f = e^{-V}$, for some convex function V.

- Properties:

  1. Show that the support of a log-concave function is convex.
  2. Show that the product of log-concave functions is log-concave.
  3. Show that if $f : C \to [0,+\infty)$ is concave on its support, where $C \subset \mathbb{R}^n$ is convex, then f is log-concave. Is a log-concave function concave?
  4. Show that if $f : \mathbb{R} \to [0,+\infty)$ is log-concave and even, then f is non-increasing on $[0,+\infty)$ and non-decreasing on $(-\infty,0]$. Deduce that f attains its maximum at 0.
  5. Show that if $f : [0,+\infty) \to [0,+\infty)$ is log-concave and attains its maximum at 0, then f is non-increasing on $[0,+\infty)$.

- Examples:

  1. Show that the Gaussian distribution is log-concave (the density is $\frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{\|x\|_2^2}{2}}$).
  2. Show that the Laplace distribution is log-concave (the density is $\frac{1}{2}e^{-|x|}$).
  3. Show that 1_K is log-concave if and only if $K \subset \mathbb{R}^n$ is convex. Deduce that if X is uniformly distributed on a convex set, then X is log-concave.
  4. Show that the $\chi^2(k)$ distribution, with $k \geq 2$, is log-concave (the density is $\frac{1}{2^{k/2}\Gamma(k/2)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} 1_{\mathbb{R}_+}(x)$).
  5. Show that the function $e^{-\frac{|x|^p}{p}}$, $p \geq 1$, is log-concave. Is the function $e^{-\frac{|x|^p}{p}}$, $p \in (0,1)$, log-concave?
  6. Is the Cauchy distribution log-concave?

Exercise 2.

Let X be an arbitrary random vector in $\mathbb{R}^n$ with mean μ and covariance matrix K_X. Prove that $Z = K_X^{-\frac{1}{2}}(X-\mu)$ is isotropic.

Exercise 3.

Let X, Y be i.i.d. isotropic log-concave random vectors in $\mathbb{R}^n$.

1. Show that $\mathbb{E}[\|X\|_2^2] = n$.
2. Show that $\mathbb{E}[\|X-Y\|_2^2] = 2n$.
3. Show that there exist universal constants $c_1, c_2 > 0$ such that

$$c_1\sqrt{n} \leq \mathbb{E}[\|X\|_2] \leq c_2\sqrt{n}.$$

In the next exercise, we slightly change the definition of isotropicity for convex body and log-concave functions. The following definition is equivalent:

- We say that a convex body $K \subset \mathbb{R}^n$ is isotropic if K is centered, if K has volume 1, and if there exists a constant $L_K > 0$ such that

$$\int_K x_i^2 \,\mathrm{d}x = L_K^2, \quad \forall i \in \{1, \dots, n\}.$$

- We say that a log-concave probability density function $f\colon \mathbb{R}^n \to [0, +\infty)$ is isotropic if f is centered, $\|f\|_\infty = 1$, and if there exists a constant $L_f > 0$ such that

$$\int_{\mathbb{R}^n} x_i^2 f(x) \,\mathrm{d}x = L_f^2, \quad \forall i \in \{1, \dots, n\}.$$

Exercise 4.

- Let $L_C > 0$ be the least number such that for all isotropic convex body $K \subset \mathbb{R}^n$,

$$\int_K x_i^2 \,\mathrm{d}x \leq L_C, \quad i = 1, \dots, n.$$

- Let $L_g > 0$ be the least number such that for all isotropic log-concave probability density function $f\colon \mathbb{R}^n \to [0, +\infty)$,

$$\int_{\mathbb{R}^n} x_i^2 f(x) \,\mathrm{d}x \leq L_g, \quad i = 1, \dots, n.$$

In other words,

$$L_C = \sup\{L_K : \ K \subset \mathbb{R}^n \text{ isotropic convex body}\},$$

$$L_g = \sup\{L_f : \ f\colon \mathbb{R}^n \to [0, +\infty) \text{ isotropic log-concave density function}\}.$$

The goal of this exercise is to show that L_C and L_g are equivalent. First, we introduce the following two lemmas:

Lemma A: Let $\lambda \in [0, 1]$. Let $g, h, m : [0, +\infty) \to [0, +\infty)$ be such that for every $r, s > 0$,

$$m\left(((1-\lambda)r^{-1} + \lambda s^{-1})^{-1}\right) \geq g(r)^{\frac{(1-\lambda)s}{(1-\lambda)s+\lambda r}} h(s)^{\frac{\lambda r}{(1-\lambda)s+\lambda r}}.$$

Then,

$$\int_0^{+\infty} m(r)\,\mathrm{d}r \geq \left((1-\lambda)\left(\int_0^{+\infty} g(r)\,\mathrm{d}r\right)^{-1} + \lambda\left(\int_0^{+\infty} h(r)\,\mathrm{d}r\right)^{-1}\right)^{-1}.$$

Lemma B: Let $F : [0,+\infty) \to [0,+\infty)$ be a non-increasing log-concave function. Then, for every $0 \leq p \leq q < +\infty$,

$$F(0)^q \Gamma(p+1)^{q+1}\left(\int_0^{+\infty} r^q F(r)\,\mathrm{d}r\right)^{p+1} \leq F(0)^p \Gamma(q+1)^{p+1}\left(\int_0^{+\infty} r^p F(r)\,\mathrm{d}r\right)^{q+1}.$$

1. Show that we always have $L_C \leq L_g$.
2. Let $f : \mathbb{R}^n \to [0,+\infty)$ be an isotropic log-concave probability density function.

   For every $p \geq 1$, we define the following function on $\mathbb{R}^n$ by

$$\|x\|_p = \begin{cases} \left(\int_0^{+\infty} p f(rx) r^{p-1}\,\mathrm{d}r\right)^{-\frac{1}{p}} & \text{if } x \in \mathbb{R}^n \setminus \{0\} \\ 0 & \text{if } x = 0 \end{cases}.$$

   a) Show that for every $\lambda \in \mathbb{R}$, for every $x \in \mathbb{R}^n$,

$$\|\lambda x\|_p = |\lambda| \|x\|_p,$$

   and that

$$\|x\|_p = 0 \Longleftrightarrow x = 0.$$

   b) Show that

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p.$$

   Hint: Define the functions $g, h, m : [0,+\infty) \to [0,+\infty)$ by

$$g(r) = f(rx)r^{p-1}, \quad h(r) = f(ry)r^{p-1}, \quad m(r) = f(r\frac{x+y}{2})r^{p-1},$$

   and apply Lemma A with $\lambda = \frac{1}{2}$ to the functions g, h, m.

3. Let $f : \mathbb{R}^n \to [0,+\infty)$ be an isotropic log-concave probability density function.

   Recall that from question 2., the function

$$\|x\|_p = \begin{cases} \left(\int_0^{+\infty} p f(rx) r^{p-1}\,\mathrm{d}r\right)^{-\frac{1}{p}} & \text{if } x \in \mathbb{R}^n \setminus \{0\} \\ 0 & \text{if } x = 0 \end{cases}.$$

   defines a norm on $\mathbb{R}^n$, for every $p \geq 1$. Hence, the unit ball of $\|\cdot\|_{n+2}$ is a symmetric convex body in $\mathbb{R}^n$, which we denote by K_{n+2}.

   a) Show that

$$\int_{\mathbb{R}^n} x_i^2 f(x)\,\mathrm{d}x = \int_{K_{n+2}} x_i^2\,\mathrm{d}x, \quad \forall i \in \{1,\ldots,n\}.$$

   Hint: Integrate in polar coordinates.

   b) Deduce that $\frac{K_{n+2}}{\mathrm{Vol}(K_{n+2})^{\frac{1}{n}}}$ is isotropic.

c) Deduce that

$$L_f^2 \leq L_C^2 \mathrm{Vol}(K_{n+2})^{1+\frac{2}{n}}.$$

Hint: Note that

$$\int_{\frac{K_{n+2}}{\mathrm{Vol}(K_{n+2})^{\frac{1}{n}}}} x_i^2 \, \mathrm{d}x = \frac{1}{\mathrm{Vol}(K_{n+2})^{1+\frac{2}{n}}} \int_{K_{n+2}} x_i^2 \, \mathrm{d}x.$$

d) Deduce that

$$nL_f^2 \leq \frac{n+2}{n^{\frac{2}{n}}} L_C^2 \left(\int_{S^{n-1}} \left(\int_0^{+\infty} f(r\theta) r^{n+1} \, \mathrm{d}r \right)^{\frac{n}{n+2}} \mathrm{d}\theta \right)^{\frac{n+2}{n}}.$$

e) Show that

$$\int_0^{+\infty} f(r\theta) r^{n+1} \, \mathrm{d}r \leq \frac{n(n+1)}{((n-1)!)^{\frac{2}{n}}} \left(\int_0^{+\infty} f(r\theta) r^{n-1} \, \mathrm{d}r \right)^{\frac{n+2}{n}}.$$

Hint: Apply Lemma B to $r \to f(r\theta)$ with $p = n-1$ and $q = n+1$.

f) Deduce that

$$L_g^2 \leq \frac{(n+1)(n+2)}{(n!)^{\frac{2}{n}}} L_C^2,$$

and hence

$$L_g \leq e L_C.$$