Homework #3 — **Concentration Inequalities**

Exercise 1. (Tails of the normal distribution)

Let $Z \sim N(0,1)$. Prove that for all $t > 0$,

$$\left(\frac{1}{t} - \frac{1}{t^3}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \leq \mathbb{P}(Z \geq t) \leq \frac{1}{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}.$$

Deduce that for all $t \geq 1$, the tail distribution is upper bounded by the density:

$$\mathbb{P}(Z \geq t) \leq \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}.$$

Exercise 2. (Truncated normal distribution)

Let $Z \sim N(0,1)$. Show that for all $t > 0$,

$$\mathbb{E}[Z^2 1_{\{Z>t\}}] = t\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} + \mathbb{P}(Z \geq t) \leq \left(t + \frac{1}{t}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}.$$

Hint: Integrate by parts.

Exercise 3. (Bounding the hyperbolic cosine)

Recall that

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Prove that for all $x \in \mathbb{R}$,

$$\cosh(x) \leq e^{\frac{x^2}{2}}.$$

Exercise 4.

Suppose we want to estimate the mean μ of a random variable X from a sample $X_1, \ldots, X_N$ drawn independently from the distribution of X. We want an ε-accurate estimate, i.e. one that falls in the interval $(\mu - \varepsilon, \mu + \varepsilon)$. Use the sample mean $\hat{\mu} = \frac{X_1 + \cdots + X_N}{N}$ to show that a sample of size $N = O(\frac{\sigma^2}{\varepsilon^2})$ is suffcient to compute an ε-accurate estimate with probability at least 3/4, where $\sigma^2 = \mathrm{Var}(X)$.

Exercise 5. (Small ball probabilities)

Let $X_1, \ldots, X_N$ be non-negative independent random variables with continuous distributions. Assume that the densities of X_i are uniformly bounded by a constant $C > 0$.

1. Prove that the moment generating function of the X_i's satisfy for all $t > 0$,

$$\mathbb{E}[e^{-tX_i}] \leq \frac{C}{t}.$$

2. Deduce that for all $\varepsilon > 0$,

$$\mathbb{P}(X_1 + \cdots + X_N \leq \varepsilon N) \leq (Ce\varepsilon)^N.$$