University of Florida **MAP 6472**

Homework #3

Exercise 1. Consider the probability space $\big([0,\frac{\pi}{2}], \mathcal{B}([0,\frac{\pi}{2}]), \text{Uniform}([0,\frac{\pi}{2}])\big)$.

1. Explain in a few lines why $X\colon \big([0,\frac{\pi}{2}], \mathcal{B}([0,\frac{\pi}{2}])\big) \to ([0,1], \mathcal{B}([0,1]))$, $\omega \mapsto \cos(\omega)$ and $Y\colon \big([0,\frac{\pi}{2}], \mathcal{B}([0,\frac{\pi}{2}])\big) \to ([0,1], \mathcal{B}([0,1]))$, $\omega \mapsto \sin(\omega)$ are random variables.
2. Find the CDF (Cumulative Distribution Function) of X and Y.
3. Find the PDF (Probability Density Function) of X and Y. What can you conclude?
4. Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ with 2 different methods.

Exercise 2. Let (X,Y) be a couple of random variables with values in $\{1,2,3\} \times \{1,2,3,4\}$ having the following distribution

X \ Y	1	2	3	4
1	0.08	0.04	0.16	0.12
2	0.04	0.02	0.08	0.06
3	0.08	0.04	0.16	0.12

1. Find the distribution of X and Y.
2. Are X and Y independent?

Exercise 3. Consider the experiment of throwing a dice with 6 sides, and let X be the number obtained. Let Y be the random variable with value 1 if X is even, and with value 0 if X is odd.

1. Find the distribution of the random vector (X,Y).
2. Are X and Y independent?
3. Compute the variance of X, Y, and $X+Y$.

Exercise 4. Let $X \sim \mathcal{N}(0,1)$ (standard Gaussian). Find the distribution of $Y = \exp(X)$.

Exercise 5. Let $X \sim \text{Unif}([0,1])$ (uniform on $[0,1]$). Fix $0 < a < 1$. Prove that the distribution of $Y = \min(X,a)$ is a mixture of a continuous random variable and a Dirac measure.

Exercise 6. Let X be a random variable. Prove that for all $p \geq 1$,

$$\mathbb{E}[|X|^p] = \int_0^{+\infty} pt^{p-1}\mathbb{P}(|X| > t)\, dt.$$

Exercise 7.

1. Let X be a random variable with finite second moment (that is $\mathbb{E}[X^2] < +\infty$). Prove that for all $\varepsilon > 0$,

$$\mathbb{P}(|X - \mathbb{E}[X]| > \varepsilon) \leq \frac{\mathrm{Var}(X)}{\varepsilon^2}.$$

2. Let $\{X_n\}$ be a sequence of independent Bernoulli distribution all with same parameter p. Define $S_n = \sum_{i=1}^{k} X_k$. Deduce from question 1) that

$$\lim_{n\to+\infty} \mathbb{P}\left(\left|\frac{S_n}{n} - p\right| > \varepsilon\right) = 0.$$