University of Florida **MAP 6473**

Solution - Homework #3

Exercise 1.

1. • S_n is $\mathcal{F}_n$-measurable, hence M_n is $\mathcal{F}_n$-measurable.

- $\mathbb{E}[|M_n|] = \mathbb{E}[|(S_n - \mathbb{E}[S_n])^2 - \mathrm{Var}(S_n)|] \leq \mathbb{E}[|S_n - \mathbb{E}[S_n]^2] + \mathrm{Var}(S_n) = 2\mathrm{Var}(S_n) < +\infty.$

- We have

$$\begin{aligned} M_{n+1} &= (S_{n+1} - \mathbb{E}[S_{n+1}])^2 - \mathrm{Var}(S_{n+1}) \\ &= (S_n - n\mathbb{E}[X_1] + X_{n+1} - \mathbb{E}[X_1])^2 - n\mathrm{Var}(X_1) - \mathrm{Var}(X_1) \\ &= M_n + (X_{n+1} - \mathbb{E}[X_1])^2 + 2(S_n - n\mathbb{E}[X_1])(X_{n+1} - \mathbb{E}[X_1]) - \mathrm{Var}(X_1). \end{aligned}$$

Hence,

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n + \mathbb{E}[(X_{n+1} - \mathbb{E}[X_1])^2|\mathcal{F}_n] + 2\mathbb{E}[(S_n - n\mathbb{E}[X_1])(X_{n+1} - \mathbb{E}[X_1])|\mathcal{F}_n] - \mathrm{Var}(X_1).$$

Note that

$$\mathbb{E}[(X_{n+1} - \mathbb{E}[X_1])^2|\mathcal{F}_n] = \mathbb{E}[(X_{n+1} - \mathbb{E}[X_1])^2] = \mathrm{Var}(X_{n+1}).$$

Also,

$$\mathbb{E}[(S_n - n\mathbb{E}[X_1])(X_{n+1} - \mathbb{E}[X_1])|\mathcal{F}_n] = (S_n - n\mathbb{E}[X_1])\,\mathbb{E}[X_{n+1} - \mathbb{E}[X_1]|\mathcal{F}_n] = 0.$$

This concludes that $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$. And thus $\{M_n\}$ is a martingale.

2. Recall that

$$\cosh(u) = \frac{e^u + e^{-u}}{2}.$$

- M_n^u is $\mathcal{F}_n$-measurable.

- We have

$$\mathbb{E}[|M_n^u|] = \frac{\mathbb{E}[e^{u\,S_n}]}{\cosh(u)^n} = \frac{\mathbb{E}[\Pi_{i=1}^n e^{u\,X_i}]}{\cosh(u)^n} = 1 < +\infty.$$

- We have

$$M_{n+1}^u = M_n \frac{e^{uX_{n+1}}}{\cosh(u)},$$

and thus

$$\mathbb{E}[M_{n+1}^u|\mathcal{F}_n] = M_n \mathbb{E}\left[\frac{e^{uX_{n+1}}}{\cosh(u)}\right] = M_n.$$

Exercise 2.

1. By assumption, for all $n \geq 1$, $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$ and $\mathbb{E}[X_n] = \mathbb{E}[X_1]$. Hence,

$$0 = \mathbb{E}[X_{n+1} - X_n] = \mathbb{E}[\mathbb{E}[X_{n+1} - X_n|\mathcal{F}_n]] = \mathbb{E}[\mathbb{E}[X_{n+1}|\mathcal{F}_n] - X_n].$$

Recall that if Y is such that $Y \geq 0$ and $\mathbb{E}[Y] = 0$, then $Y = 0$. We deduce that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] - X_n = 0$.

2. We easily check that

$$\mathbb{E}[X_{n+1} \vee a | \mathcal{F}_n] \geq X_n \vee a.$$

Hence, $\{X_n \vee a\}$ is a sub-martingale. From 1., it is also a martingale since the X_i's are i.i.d.

Exercise 3. (**Wald's identity**)

1. We have

$$\mathbb{E}[S_\tau] = \mathbb{E}[S_\tau \sum_{n\geq 1} 1_{\{\tau=n\}}] = \mathbb{E}[\sum_{n\geq 1} S_n 1_{\{\tau=n\}}]$$

$$= \mathbb{E}[\sum_{n\geq 1}\sum_{k=1}^{n} X_k 1_{\{\tau=n\}}] = \mathbb{E}[\sum_{k\geq 1}\sum_{n\geq k} X_k 1_{\{\tau=n\}}] = \mathbb{E}[\sum_{k\geq 1} X_k 1_{\{\tau\geq k\}}].$$

By Fubini,

$$\mathbb{E}[\sum_{k\geq 1} X_k 1_{\{\tau\geq k\}}] = \sum_{k\geq 1} \mathbb{E}[X_k 1_{\{\tau\geq k\}}] = \sum_{k\geq 1} \mathbb{E}[\mathbb{E}[X_k 1_{\{\tau\geq k\}} | \mathcal{F}_{k-1}]] = \sum_{k\geq 1} \mathbb{E}[1_{\{\tau\geq k\}} \mathbb{E}[X_k | \mathcal{F}_{k-1}]]$$

$$= \sum_{k\geq 1} \mathbb{E}[1_{\{\tau\geq k\}} \mathbb{E}[X_k]] = \mathbb{E}[X_1] \sum_{k\geq 1} \mathbb{P}(\tau \geq k) = \mathbb{E}[X_1]\mathbb{E}[\tau],$$

where the last identity comes from the fact that for all random variable X with values in $\mathbb{N}$,

$$\sum_{k\geq 1} \mathbb{P}(X \geq k) = \mathbb{E}[X] \qquad \text{(Exercise).}$$

2. Consider $\tau \wedge n$, for all $n \geq 1$. Note that $\tau \wedge n$ is increasing in n and is non-negative, and $\tau \wedge n \to \tau$ a.s. Thus, by monotone convergence,

$$\lim_n \mathbb{E}[\tau \wedge n] = \mathbb{E}[\tau].$$

Since $\tau \wedge n$ is integrable (bounded by n), by Wald's identity (question 1.),

$$\mathbb{E}[S_{\tau\wedge n}] = \mathbb{E}[X_1]\mathbb{E}[\tau \wedge n].$$

By assumption, $\mathbb{E}[X_1] \neq 0$ and $\sup_n \mathbb{E}[S_{\tau\wedge n}] < +\infty$, hence

$$\mathbb{E}[\tau \wedge n] = \frac{\mathbb{E}[S_{\tau\wedge n}]}{\mathbb{E}[X_1]} \leq \frac{\sup_n \mathbb{E}[S_{\tau\wedge n}]}{\mathbb{E}[X_1]} < +\infty.$$

By letting n goes to $+\infty$, we conclude that $\mathbb{E}[\tau] \leq \frac{\sup_n \mathbb{E}[S_{\tau\wedge n}]}{\mathbb{E}[X_1]} < +\infty$.

Exercise 4.

By contradiction, assume that τ is integrable. Then, by Wald's identity,

$$\mathbb{E}[S_\tau] = \mathbb{E}[X_1]\mathbb{E}[\tau].$$

But $\mathbb{E}[X_1] = 0$ and $S_\tau = 1$ implies $\mathbb{E}[S_\tau] = 1$. Contradiction.

Exercise 5. (**Doob's inequality**)

Done in class.