Homework #3 - Markov Chains

Exercise 1.

Among the following matrices, which ones are transition matrices of a Markov chain? For each transition matrix, draw the corresponding diagram.

$$A = \begin{pmatrix} 1/4 & 2/3 \\ 3/4 & 1/3 \end{pmatrix}; \quad B = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 3/4 & 1/2 & -1/4 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}; \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 1/6 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}; \quad D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Exercise 2.

3 yellow and 3 green balls are distributed in two urns in such a way that each urn contains 3 balls. We say that the system is in state i, $i = 0, 1, 2, 3$, if the first urn contains i yellow balls. At each step, we draw at random one ball from each urn and place the ball drawn from the first urn to the second urn, and conversely with the ball from the second urn. We assume that the urns are independent. Let X_n denote the state of the system after the n-th step. Calculate the transition matrix of $\{X_n\}$.

Exercise 3.

Let $\{X_n\}$ be a homogeneous Markov chains on a space of states $\{1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.8 & 0.2 & 0 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}.$$

Compute the following probability:

1. $\mathbb{P}(X_1 = 2, X_2 = 3, X_3 = 1, X_4 = 3, X_5 = 1, X_6 = 3, X_7 = 2 | X_0 = 3)$.
2. $\mathbb{P}(X_{n+2} = 3 | X_n = 2)$, for $n \geq 0$.
3. $\mathbb{P}(X_{n-1} = 1, X_{n+1} = 1 | X_{n-2} = 1)$, for $n \geq 2$.

Exercise 4.

Let $\{X_n\}$ be a homogeneous Markov chains on a space of states $\{1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

If $\mathbb{P}(X_0 = 1) = \mathbb{P}(X_0 = 2) = \frac{1}{4}$, find $\mathbb{E}[X_1]$.

Exercise 5.

Let $\{X_n\}$ be a homogeneous Markov chains on a space of states $\{0, 1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Determine which states are transient and which are recurrent.

Exercise 6.

Consider the following Markov chains $\{X_n\}$ on the state space $E = \{0,1,2,3,4,5\}$ with transition matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 1/5 & 2/5 & 1/5 & 0 & 1/5 \\ 0 & 0 & 0 & 1/6 & 1/3 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/4 & 0 & 3/4 \end{pmatrix}$$

1. Draw the corresponding diagram.
2. Classify the states of $\{X_n\}$ (recurrent, transient, absorbing).
3. For $C \subset E$, Define $T_C = \min\{n \geq 0 : X_n \in C\}$, and for $x \in E$, define $\rho_{x,C} = \mathbb{P}_x(T_C < +\infty)$. Compute $\rho_{1,0}$ and $\rho_{2,0}$.

Exercise 7.

Remark: The invariant measure (also called stationary measure) can be interpreted as the long-run proportion of time the Markov chain spend in each state.

Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability $\alpha \in (0,1)$; and if it does not rain today, then it will rain tomorrow with probability $\beta \in (0,1)$.

1. Find the transition matrix associated with this experiment.
2. Draw the corresponding diagram.
3. Explain why this Markov chain is recurrent.
4. Find the long-run proportion of time it is raining (that is, find the invariant measure).