Homework #4 - Applications of Markov Chains

Exercise 1. [Reliability]

An appliance has 3 states: working properly (State 1), working with defect (State 2), breakdown (State 3). The state of the appliance at time n is denoted by X_n. We model the sequence $\{X_n\}$ as a homogeneous Markov chains with transition matrix

$$P = \begin{pmatrix} r_1 & p_1 & q_1 \\ q_2 & r_2 & p_2 \\ p_3 & q_3 & r_3 \end{pmatrix}.$$

We assume moreover that $q_1 > 0$ and $p_2 > 0$.

We would like to compute the reliability at time n and the average working time of the appliance when the initial state is 1 (working properly). Formally, the goal is to compute

$$\mathbb{P}_1(S > n) = \mathbb{P}(S > n | X_0 = 1) \quad \text{and} \quad \mathbb{E}_1[S],$$

where S is the first time the appliance breaks down,

$$S = \min\{n \geq 0 : X_n = 3\}.$$

Denote, for $n \geq 0$, for $i \in \{1, 2\}$,

$$u_i(n) = \mathbb{P}_i(S > n).$$

1. Compute $u_1(0), u_2(0), u_1(1), u_2(1)$.
2. For $n \geq 1$, express $u_1(n)$ and $u_2(n)$ in terms of $u_1(n-1)$ and $u_2(n-1)$. Deduce a way to compute $u_1(n)$ and $u_2(n)$.
3. Show that for $i \in \{1, 2\}$,
$$\mathbb{E}_i[S] = \sum_{n \geq 0} \mathbb{P}_i(S > n).$$
Deduce $\mathbb{E}_1[S]$.

Exercise 2. [The gambler's ruin]

Let $\{X_n\}$ be a homogeneous Markov chains in the state space $\mathbb{N}$ and transition p. Let $0 \leq a < b$ be two natural numbers. Denote, for $x \in \mathbb{N}$,

$$T_x = \min\{n \geq 0 : X_n = x\}.$$

Denote also, for $y \in \mathbb{N}$,

$$u(y) = \mathbb{P}_y(T_a < T_b).$$

1. Compute $u(a)$ and $u(b)$.
2. Show that for all $y \in \mathbb{N} \setminus \{a, b\}$,
$$u(y) = \sum_{z \in \mathbb{N}} u(z) p(y, z)$$

3. Let $p \in (0,1)$. Assume that $\{X_n\}$ has transition function

$$p(x,y) = \begin{cases} p & \text{if } y = x+1 \\ 1-p & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases}.$$

Show that for all $a < y < b$,

$$u(y) = p\,u(y+1) + (1-p)\,u(y-1).$$

4. Show that for all $a < y < b$,

$$u(y+1) - u(y) = \left(\frac{1-p}{p}\right)^{y-a} (u(a+1) - u(a)).$$

5. Deduce that for all $a < y < b$,

$$\mathbb{P}_y(T_a < T_b) = q^{y-a}\frac{1-q^{b-y}}{1-q^{b-a}}, \qquad \text{where } q = \frac{1-p}{p}.$$

6. **Application:** At the casino, a gambler starts with \$10 and plays at a slot machine. Each turn, the gambler wins \$1 with probability 9/19 and loses \$1 with probability 10/19. The gambler decides to stop playing either when she/he reaches \$25 either when she/he has no money left. What is the probability that the gambler ends up winning at the slot machine?

Exercise 3. [Ehrenfest model]

Consider N particles contained in two boxes A and B. At each time, one particle is taken at random from one of the boxes and is put into the other box. Denoting X_n the number of particles in A at time n, one may consider $\{X_n\}$ as a homogeneous Markov chains with state space in $E = \{0, 1, \dots, N\}$ and transition matrix

$$p(i,j) = \begin{cases} \frac{N-i}{N} & \text{if } j = i+1 \\ \frac{i}{N} & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases}.$$

1. Justify in a few words why $\{X_n\}$ has an invariant probability.

2. A probability π on a space state E is said to be <u>**reversible**</u> if for all $i, j \in E$,

$$\pi(i)p(i,j) = \pi(j)p(j,i).$$

Prove that a reversible probability is an invariant probability.

3. Find the invariant probability of $\{X_n\}$.
<u>**Hint:**</u> Find the reversible probability.