University of Florida **MAP 6467**

Homework #4 – **Stochastic Integration and Ito's formula**

Exercise 1.
Suppose A_t is a simple process with $|A_t| \leq C$ for all t. Let

$$Z_t = \int_0^t A_s \, dB_s.$$

Show that $\mathbb{E}[Z_t^4] \leq 4C^4t^2$.

Hint: Write Z_t explicitly and expand 4th power.

Exercise 2.
Use Itô's formula to show that $X_t = f(t, B_t)$ is an Ito process, where

1. $f(t, x) = \sin(x)$.
2. $f(t, x) = e^{-t}(tx)^2$.

Exercise 3.
Suppose an asset follows the following geometric SDE

$$dX_t = 4X_t dt + X_t dB_t.$$

1. Write the exact solution of this equation. (find X_t as a function of B_t.)
2. Suppose $X_0 = 2$. What is the probability that $X_1 > 8$?
3. Suppose $X_0 = 1$. What is the probability that $X_2 < 6$?
4. Suppose $X_0 = 4.4565$. What is the probability that $X_t < 0$ for some $2 < t < 5$?

Exercise 4.
Let $\{B_t\}$ be a standard Brownian motion. Suppose that two assets X_t, Y_t follow the SDEs

$$dX_t = X_t[\mu_1 dt + \sigma_1 dB_t],$$

$$dY_t = Y_t[\mu_2 dt + \sigma_2 dB_t].$$

Suppose also that $X_0 = Y_0 = 1$.

1. Let $Z_t = X_t Y_t$. Give the SDE satisfied by Z_t.
2. Does there exist a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(X_t) = B_t$ for all t?
3. Does there exist a function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(Z_t) = B_t$ for all t?

Exercise 5.

Suppose that X_t satisfies the SDE

$$dX_t = X_t \left(\frac{1}{2} dt + dB_t \right), \quad X_0 = 2.$$

Let $M = \max_{0 \le t \le 1} X_t$.

1. Find the density of M.
2. Find the probability that $M \ge 4$.