Homework #4 — **Entropy**

Recall that for $p \geq 0$, the p-th **Rényi entropy** of $X \sim f_X$ is defined by

$$h_p(X) \triangleq \frac{1}{1-p} \log\left(\int f_X^p\right).$$

The Rényi entropy of order 0, 1, and ∞ are defined by taking limits (that is, by letting p goes to 0, 1, ∞). Define the **Rényi entropy power** of X by

$$N_p(X) \triangleq e^{\frac{2}{n} h_p(X)}.$$

Exercise 1.

Compute the Rényi entropy of the Gaussian, uniform, and exponential distributions.

Exercise 2. (Chain Rule for Entropy)

Let $X_1, \ldots, X_k$ be random vectors with finite Shannon entropy.

1. Prove that
$$h(X_1, \ldots, X_k) = \sum_{i=1}^{k} h(X_i | X_1, \ldots, X_{i-1}).$$
2. Deduce that $h(X_1, \ldots, X_k) \leq \sum_{i=1}^{k} h(X_i)$.
3. Deduce that if the X_i's are independent, then $h(X_1, \ldots, X_k) = \sum_{i=1}^{k} h(X_i)$.

Exercise 3.

1. Show that
$$h_0(X) = \log(\mathrm{Vol}(\mathrm{supp}(X))),$$
$$h_1(X) = h(X) = -\int f_X \log(f_X), \quad (\textbf{Shannon entropy})$$
$$h_\infty(X) = -\log(\|f_X\|_\infty).$$
2. Show that for every $p > 0$, for every $\lambda \in \mathbb{R}$,
$$N_p(\lambda X) = \lambda^2 N_p(X).$$

Exercise 4.

1. a) Show that for every $p > 0$,

$$h_p(X) \geq -\log(\|f_X\|_\infty).$$

b) For what distribution is there equality?

Hint: Consider uniform distributions.

c) More generally, show that for every $p \leq q$,

$$h_p(X) \geq h_q(X).$$

2. Let $X \sim f_X$ be a log-concave random vector.

a) Using the definition of log-concavity, show that for every $x, y \in \mathbb{R}^n$, for every $\lambda \in (0, 1)$,

$$f_X((1-\lambda)x + \lambda y)^{\frac{1}{\lambda}} \geq f_X(x)^{\frac{1-\lambda}{\lambda}} f_X(y). \qquad (\star)$$

b) Integrating over y in inequality $(\star)$, deduce that

$$\frac{1}{\lambda^n} \int f_X(u)^{\frac{1}{\lambda}}\,\mathrm{d}u \geq f_X(x)^{\frac{1-\lambda}{\lambda}}. \qquad (\star\star)$$

c) Taking supremum over all x in inequality $(\star\star)$, and setting $p = \frac{1}{\lambda}$, deduce that

$$\int f_X^p \geq \frac{1}{p^n} \|f_X\|_\infty^{p-1}.$$

d) Deduce that

$$h_p(X) \leq \frac{n}{p-1} \log(p) - \log(\|f_X\|_\infty).$$

e) For what distribution is there equality?

Hint: Consider exponential distributions.

f) Conclude that Rényi entropies are equivalent for log-concave random vector X:

$$N_\infty(X) \leq N_p(X) \leq p^{\frac{2}{p-1}} N_\infty(X).$$

g) Deduce that for log-concave X, we have

$$N_\infty(X) \leq N(X) \leq e^2 N_\infty(X).$$