Homework #4

Exercise 1. Let $\{X_n\}$ be a sequence of independent random variables such that

$$\mathbb{P}(X_n = -n) = \mathbb{P}(X_n = n) = \frac{1}{2n\log(n)}, \qquad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n\log(n)}.$$

1. Determine the modes of convergence of $\{X_n\}$.
2. Compute $\mathbb{E}[X_n]$ and $\mathrm{Var}(X_n)$. Deduce that $\frac{X_1+\cdots+X_n}{n}$ converges to 0 in L^2.
3. Prove that $\frac{X_1+\cdots+X_n}{n}$ does not converge almost surely.

Exercise 2. Let $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), \ldots, X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ be Gaussians.
Find the distribution of $X_1 + \cdots + X_n$.

Exercise 3. Let X be a standard Gaussian $\mathcal{N}(0,1)$, and let ε be a symmetric Bernoulli distribution, that is

$$\mathbb{P}(\varepsilon = -1) = \mathbb{P}(\varepsilon = 1) = \frac{1}{2}.$$

1. Prove that εX is a Gaussian random variable.
2. Is $(X, \varepsilon X)$ a Gaussian random vector?
3. Prove that X and εX are uncorrelated.
4. Are X and εX independent?

Exercise 4. Let $p \in (0,1)$. Let $\{X_n\}$ be a sequence of i.i.d. random variables with distribution

$$\mathbb{P}(X_n = 1) = p, \quad \mathbb{P}(X_n = -1) = 1 - p.$$

Let us define, for $n \geq 1$, $Y_n = \prod_{i=1}^n X_i$.

1. Compute $\mathbb{E}[Y_n]$. Deduce the distribution of Y_n.
2. Does $\{Y_n\}$ converge in distribution?

Exercise 5. Let $\theta > 0$. Let $\{X_n\}$ be a sequence of independent random variables such that for all $n \geq 1$, X_n has a geometric distribution of parameter $\frac{\theta}{n}$.
Prove that $\{\frac{X_n}{n}\}$ converges in distribution to an exponential distribution of parameter θ.

Exercise 6. Let $\lambda > 0$. Let $\{X_n\}$ be a sequence of random variables such that for all $n \geq 1$, X_n is a Binomial distribution $\mathcal{B}(n, \frac{\lambda}{n})$.
Prove that $\{X_n\}$ converges in distribution to a Poisson distribution of parameter λ.

Exercise 7. Let $\{X_n\}$ be a sequence of random variables.

1. Assume that for all $n \geq 1$, $\mathbb{P}(X_n = \frac{1}{n}) = 1$. Prove that $\{X_n\}$ converges in distribution to some random variable X. Does F_{X_n} converges to F_X pointwise?

2. Assume that for all $n \geq 1$, $\mathbb{P}(X_n = n) = 1$. Does F_{X_n} converges pointwise to a CDF?

Exercise 8. Let $\{(X_n, Y_n)\}$ be a sequence of random vectors such that for all n, X_n and Y_n are independent. Assume that X_n converges to X in distribution and Y_n converges to Y in distribution, with $X \perp\!\!\!\perp Y$.

1. Prove that $\{(X_n, Y_n)\}$ converges to (X, Y) in distribution.

2. Deduce that, in this case, $\{X_n + Y_n\}$ converges to $X + Y$ in distribution, and that $\{X_n Y_n\}$ converges to XY in distribution.

3. Does the result hold without the assumption of independence?

Exercise 9. Consider an urn containing 2 green balls and 4 red balls. At each turn, we take a ball from the urn at random and put it back. At the i-th turn, we associate a random variable Y_i defined as

$$Y_i = 1 \text{ if the ball is green, and } Y_i = 0 \text{ if the ball is red.}$$

Define for $n \geq 1$,

$$X_n = \frac{1}{n} \sum_{i=1}^{n} Y_i.$$

1. Prove that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\mathbb{P}\left(\left|X_n - \frac{1}{3}\right| \geq 0.02\right) \leq 0.01.$$

2. Estimate n_0.