Homework #5 - **Brownian Motion**

Exercise 1.

We model the price of an asset as a Brownian motion with value B_t at time t. Suppose we are allowed to trade our asset only at the following times: $0 = t_0 < t_1 < \cdots < t_n = 1$. At time t_k we can choose to hold X_k shares of our asset, and we must hold these shares up until the next time period t_{k+1}.

1. What is the value of our portfolio between time t_{k-1} and t_k?
2. Express a formula describing the change in our wealth over the time period $[0, 1]$.
3. Strictly speaking, can the price of an asset be modeled by a Brownian motion? Why?

Exercise 2.

Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion. Compute the following probabilities:

1. $\mathbb{P}(B_1 = 0)$.
2. $\mathbb{P}(B_2 \geq 0)$.
3. $\mathbb{P}(B_2 \geq 0, B_1 \leq 0)$.
4. $\mathbb{P}(B_0 = 0)$.

Exercise 3.

Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion.

1. Are B_1 and B_2 independent?
2. Are $B_3 - B_1$ and $B_{\frac{2}{5}}$ independent?
3. Are $B_5 - B_3$ and B_4 independent?
4. Are B_2 and B_0 independent?

Exercise 4.

What is the distribution of $B_s + B_t$, for $s \leq t$.

Exercise 5.

Compute $\mathbb{E}[B_t B_s B_r]$, for $r \leq s \leq t$.

Exercise 6.

Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion. Prove that

$$\text{Covariance}(B_t, B_s) = \min(s, t),$$

where the covariance between two random variables X and Y is defined as

$$\text{Covariance}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$

Exercise 7.

Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion. Define, for $x \in \mathbb{R}$,

$$T_x = \inf\{t > 0 : B_t = x\}.$$

Compute $\mathbb{P}(T_1 < T_{-1} < T_2)$.