University of Florida **MAP 6473**

Homework #4 – **Brownian Motion**

Exercise 1.

Let $\{B_t\}$ be a standard Brownian motion. Prove that

$$\text{Cov}(B_t, B_s) = \min(t, s).$$

Exercise 2.

Prove that a Brownian motion $\{B_t\}$ is a continuous-time martingale (with respect to the same filtration).

Exercise 3.

Let Z be a standard Gaussian. Define, for $t \geq 0$, $X_t = \sqrt{t}Z$.

1. Prove that $\{X_t\}$ has almost surely continuous paths, and that $X_t \sim \mathcal{N}(0, t)$.
2. Is $\{X_t\}$ a Brownian motion?

Exercise 4.

Let $\{B_t\}$ and $\{\widetilde{B}_t\}$ be two independent standard Brownian motion. Let $\rho \in (0, 1)$. Define, for $t \geq 0$,

$$X_t = \rho B_t + \sqrt{1 - \rho^2}\widetilde{B}_t.$$

Is $\{X_t\}$ a Brownian motion?

Exercise 5. (Brownian Bridge)

A stochastic process $\{X_t\}_{t \in [0,1]}$ is called **Brownian bridge** if:

i) $X_0 = X_1$.

ii) $\{X_t\}$ is a centered Gaussian process, that is, for all $t_1 < \cdots < t_n$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ is a multivariate Gaussian with mean 0.

iii) $\text{Cov}(X_t, X_s) = \min(s, t) - st$.

iv) Almost surely, $\{X_t\}$ has continuous paths.

Let $\{B_t\}$ be a standard Brownian motion and $\{X_t\}$ be a Brownian bridge.

1. Define, for $t \in [0, 1]$, $\widetilde{X}_t = B_t - tB_1$. Show that $\{\widetilde{X}_t\}$ is a Brownian bridge.
2. Let Z be a standard Gaussian. Show that $\{\widetilde{B}_t\} = X_t + tZ$, is a Brownian motion for $t \in [0, 1]$.
3. Prove that $W_t = (t + 1)X_{\frac{t}{t+1}}$ is a Brownian motion for $t \in [0, +\infty)$.

Exercise 6.

Let $\{B_t\}$ be a Brownian motion. Compute:

1. $\mathbb{P}(B_1 \geq 0)$.
2. $\mathbb{P}(B_2 \geq 0, B_1 \geq 0)$.
3. $\mathbb{P}(B_3 \geq 0, B_2 \leq 0, B_1 \leq 0)$.

Exercise 7.

Let $\{B_t\}$ be a Brownian motion. Define

$$T = \min\{t \geq 0 : |B_t| = 1\}.$$

1. Define, for $n \geq 0$,
$$A_n = \{B_{n+1} - B_n > 2\}.$$
Prove that $\{A_n\}$ is a sequence of independent events such that $\sum \mathbb{P}(A_n) = +\infty$.
2. Deduce that T is finite almost surely (that is, $\mathbb{P}(T < +\infty) = 1$).