Exercise 1.
Let \( \{B_t\} \) be a standard Brownian motion. Prove that
\[
\text{Cov}(B_t, B_s) = \min(t, s).
\]

Exercise 2.
Prove that a Brownian motion \( \{B_t\} \) is a continuous-time martingale (with respect to the same filtration).

Exercise 3.
Let \( Z \) be a standard Gaussian. Define, for \( t \geq 0 \),
\[
X_t = \sqrt{t}Z.
\]
1. Prove that \( \{X_t\} \) has almost surely continuous paths, and that \( X_t \sim \mathcal{N}(0, t) \).
2. Is \( \{X_t\} \) a Brownian motion?

Exercise 4.
Let \( \{B_t\} \) and \( \{\tilde{B}_t\} \) be two independent standard Brownian motion. Let \( \rho \in (0, 1) \). Define, for \( t \geq 0 \),
\[
X_t = \rho B_t + \sqrt{1 - \rho^2} \tilde{B}_t.
\]
Is \( \{X_t\} \) a Brownian motion?

Exercise 5. (Brownian Bridge)
A stochastic process \( \{X_t\}_{t \in [0,1]} \) is called Brownian bridge if:

i) \( X_0 = X_1 \).

ii) \( \{X_t\} \) is a centered Gaussian process, that is, for all \( t_1 < \cdots < t_n \) the random vector \( (X_{t_1}, \ldots, X_{t_n}) \) is a multivariate Gaussian with mean 0.

iii) \( \text{Cov}(X_t, X_s) = \min(s, t) - st \).

iv) Almost surely, \( \{X_t\} \) has continuous paths.

Let \( \{B_t\} \) be a standard Brownian motion and \( \{X_t\} \) be a Brownian bridge.

1. Define, for \( t \in [0,1] \), \( \tilde{X}_t = B_t - tB_1 \). Show that \( \{\tilde{X}_t\} \) is a Brownian bridge.

2. Let \( Z \) be a standard Gaussian. Show that \( \{\tilde{B}_t\} = X_t + tZ, \) is a Brownian motion for \( t \in [0,1] \).

3. Prove that \( W_t = (t + 1)X_{t+1} \) is a Brownian motion for \( t \in [0, +\infty) \).
Exercise 6.
Let \( \{B_t\} \) be a Brownian motion. Compute:

1. \( \mathbb{P}(B_1 \geq 0) \).
2. \( \mathbb{P}(B_2 \geq 0, B_1 \geq 0) \).
3. \( \mathbb{P}(B_3 \geq 0, B_2 \leq 0, B_1 \leq 0) \).

Exercise 7.
Let \( \{B_t\} \) be a Brownian motion. Define
\[
T = \min\{t \geq 0 : |B_t| = 1\}.
\]

1. Define, for \( n \geq 0 \),
\[
A_n = \{B_{n+1} - B_n > 2\}.
\]
   Prove that \( \{A_n\} \) is a sequence of independent events such that \( \sum \mathbb{P}(A_n) = +\infty \).

2. Deduce that \( T \) is finite almost surely (that is, \( \mathbb{P}(T < +\infty) = 1 \)).