Lemma (Doob's upcrossing)

Let $\{X_n\}$ super martingale

Let $U_N[a,b]$ be the number of upcrossing of $[a,b]$ up to time N with $a < b$. Then

$$(b-a)\,\mathbb{E}\left[U_N[a,b]\right] \le \mathbb{E}[(X_N - a)^-]$$

Pf: (Step 1) Setup

If $U_N[a,b] = k$ then $\exists\, 0 \le s_1 < t_1 < \cdots < s_k < t_k \le N$ s.t.

$$X_{s_i} < a < b < X_{t_i}, \quad \forall i \in \{1,\dots,k\}$$

Define
$$c_1 = \begin{cases} 1 & X_0 < a \\ 0 & \text{o.w.} \end{cases}$$
$$\vdots$$
$$c_n = \begin{cases} 1 & \text{if } c_{n-1} = 1 \text{ \& } X_{n-1} \le b \\ 1 & \text{if } c_{n-1} = 0 \text{ \& } X_{n-1} < a \\ 0 & \text{o.w.} \end{cases}$$

By construction $\{c_n\}$ is $\{\mathcal{F}_n\}$-predictable where $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$

($\forall n$, c_n is $\mathcal{F}_{n-1}$-meas)

- If $x_0 < a$, then $c_1 = 1$

  & then c_n remains 1 until the first time x_n exceeds b, & then becomes 0.

  It remains 0 until x_n is smaller than a

  & then becomes 1 ...

- Assume WLOG $x_0 < a$

  By construction c_n predictable, & $c_n \in \{0,1\}$

  Define $Y_n = \sum_{j=1}^{n} c_j (x_j - x_{j-1}), \ n \geq 1$

<u>Step 2</u>: $\left[Y_N \geq (b-a)\, U_N[a,b] - (x_N - a)_-\right]$

$$Y_n = \sum_{j=1}^{n} c_j (x_j - x_{j-1}), \ n \geq 1$$

$$= \sum_{j=s_1+1}^{t_1} c_j (x_j - x_{j-1})$$

$$+ \sum_{j=s_2+1}^{t_2} c_j (x_j - x_{j-1})$$

$$+ \dots + \sum_{j=s_k+1}^{t_k} \text{''} + \sum_{j=s_{k+1}+1}^{N} \text{''}$$

$$= X_{t_1} - X_{s_1} + \dots + X_{t_k} - X_{s_k} + \sum_{j=s_{k+1}+1}^{N} \cdot$$

$$\geq (b-a) U_N[a,b) + \sum_{j=s_{k+1}+1}^{N} \cdot$$

- If $s_{k+1} \geq N$ then $X_N \geq a \Rightarrow$

$$\sum_{j=s_{k+1}+1}^{N} c_j (X_j - X_{j-1}) = 0 = (X_N - a)_-$$

- If $s_{k+1} < N$ then

$$\sum_{j=s_{k+1}}^{N} c_j (X_j - X_{j-1}) = X_N - X_{s_{k+1}}$$

Case 1: If $X_N \geq X_{s_{k+1}}$ then $X_N - X_{s_{k+1}} \geq 0$

$$\geq (X_N - a)_-$$

Case 2 : If $X_N < X_{s_{k+1}}$ then $X_N < a$

$(X_{s_{k+1}} < a)$

$\Rightarrow \; X_N - X_{S_{k+1}} \geq X_N - a = -(X_N - a)_-$

In all cases

$$\sum_{j=S_{k+1}+1}^{N} c_j (x_j - x_{j-1}) \geq -(X_N - a)_-$$

Finally $\quad Y_N \geq (b-a)\, U_N[a,b] - (X_N - a)_-$

<u>Step 3:</u> $\{Y_N\}$ is super martingale

b/c

$$* \quad \mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\left[\sum_{j=1}^{n+1} c_j (X_j - X_{j-1}) \mid \mathcal{F}_n\right]$$

$$= Y_n + \mathbb{E}[c_{n+1}(X_{n+1} - X_n \mid \mathcal{F}_n]$$

$$= Y_n + \underbrace{c_{n+1}}_{\geq 0} \underbrace{\mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n]}_{\leq 0}$$

$$\leq Y_n$$

$\Rightarrow \{Y_n\}$ super martingale

$\Rightarrow \; \mathbb{E}[Y_N] \leq \mathbb{E}[Y_1] = \mathbb{E}[c_1(X_1 - X_0)]$

$$= \mathbb{E}[X_1 - X_0] \leq 0$$

$\{X_n\}$ super mart

$$\Rightarrow \quad (b-a)\,\mathbb{E}[U_N[a,b]] \le \mathbb{E}[(X_N-a)_-]$$