1 Brownian motion

Recall that a filtered space \((\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})\) is given.

1.1 Definitions

**Definition 1.1** (Standard Brownian motion). A continuous-time stochastic process \(\{B_t\}_{t \in [0, +\infty)}\) is a standard Brownian motion if

1. \(B_0 = 0\) a.s.
2. (Stationary Gaussian increments) \(\forall 0 \leq s < t, \ B_t - B_s \sim B_t - B_s - B_0 \sim \mathcal{N}(0, t-s)\) (Gaussian of mean 0 and variance \(t-s\)).
3. (Independent increments) \(\forall 0 \leq s < t, \ B_t - B_s\) is independent of \(\mathcal{F}_s\).
4. With probability 1, the trajectories are continuous. Precisely:

\[ \exists A \subset \Omega, \mathbb{P}(A) = 1, \forall \omega \in \Omega, t \mapsto B_t(\omega) \text{ is continuous on } [0, +\infty). \]

**Remark 1.2.** One may ask whether all the assumptions are necessary in the definition of the Brownian motion. Or, in other words, does one or several assumptions imply another one.

- The continuity assumption is a necessity. To see this, let \(\{B_t\}\) be a Brownian motion and let \(U\) be uniformly distributed on \([0, 1]\). Define, for \(\omega \in \Omega\) and \(t \geq 0\),

\[ \tilde{B}_t(\omega) = B_t(\omega)1_{\{t \neq U(\omega)\}} + (1 + B_t(\omega))1_{\{t = U(\omega)\}}. \]

In this case, for all \(t \geq 0\), \(\mathbb{P}(\tilde{B}_t = B_t) = 1\), and hence \(\tilde{B}_t\) satisfies properties 1-3. of the definition. However, for all \(\omega \in \Omega\), \(t \mapsto \tilde{B}_t(\omega)\) is discontinuous (at \(t = U(\omega)\)).

- It can be shown that if 3-4 and stationary increments hold, then necessarily the increments are Gaussian.

- Property 1. is just a normalization. A brownian motion can start at any point.

- We will always consider the natural filtration \(\mathcal{F}_t = \sigma(B_s : s \leq t)\).

**Model:** Brownian motions are used to model the trajectories of small particles in a fluid, or the evolution of the stock market. Generally speaking, it is used to model erratic motions.

**Remark 1.3.** When we say “Let \(\{B_t\}_{t \geq 0}\) be a Brownian motion”, we implicitly assume the existence of a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a family of random variables \(\{B_t\}\) on \((\Omega, \mathcal{F})\) such that \(\mathbb{P}\) makes \(\{B_t\}\) a Brownian motion (that is, such that \(\{B_t\}\) satisfies the definitions 1-4. with respect to \(\mathbb{P}\)).

**Question:** Does such a probability space exist?

**Answer:** Yes, but technical to prove. This is the goal of the next section.
1.2 Construction of the Brownian motion

We will restrict the construction to \([0, 1]\). For \(n \geq 0\), denote
\[
D_n = \left\{ \frac{k}{2^n} : k \in\{0, \ldots, 2^n\} \right\}.
\]
For example,
\[
D_0 = \{0, 1\}, \quad D_1 = \left\{ 0, \frac{1}{2}, 1 \right\}, \quad D_2 = \left\{ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \right\}.
\]
Denote
\[
D = \bigcup_{n \geq 0} D_n,
\]
the dyadic of \([0, 1]\). Before starting, first note that \(D\) is dense in \([0, 1]\), and that \(\{D_n\}\) is increasing \((D_n \subset D_{n+1})\).

The process will follow the following steps:

**Step 1:** For each \(n \in \mathbb{N}\), build a continuous process \(\{B_t^{(n)}\}_{t \in [0, 1]}\) that satisfies the properties of the Brownian motion on \(D_n\).

**Step 2:** With probability 1, \(t \mapsto B_t^{(n)}\) converges uniformly on \([0, 1]\).

**Step 3:** \(\lim_{n \to +\infty} B_t^{(n)}\) is a Brownian motion.

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**Step 1: [Construction on the dyadic]**

Let \(\{Z_q\}_{q \in D}\) be a sequence of i.i.d. standard Gaussian. In particular, for all \(q \neq r \in D\), \(Z_q\) is independent of \(Z_r\), and \(Z_q \sim \mathcal{N}(0, 1)\).

**Main Lemma:** If \(X, Y\) are i.i.d. \(\mathcal{N}(0, 1)\), then \(\frac{X+Y}{\sqrt{2}}\) and \(\frac{X-Y}{\sqrt{2}}\) are i.i.d. \(\mathcal{N}(0, 1)\).

**Proof:** Exercise.

For each \(\omega \in \Omega\), we are going to build \(B_t^{(n)}(\omega)\) by induction on \(n \in \mathbb{N}\), for \(t \in D_n\), and then interpolate linearly. We drop the variable \(\omega\) next.

**For \(n = 0\):**
Set \(B_0^{(0)} = 0\) and \(B_1^{(0)} = Z_1\). Then, we interpolate linearly between \(B_0^{(0)}\) and \(B_1^{(0)}\):
\[
B_t^{(0)} = (1-t)B_0^{(0)} + tB_1^{(0)} = tZ_1, \quad t \in [0, 1].
\]

**For \(n = 1\):**
Set
\[
B_0^{(1)} = B_0^{(0)} = 0, \quad B_1^{(1)} = B_1^{(0)} = Z_1, \quad B_{\frac{1}{2}}^{(1)} = \frac{1}{2} \left( B_0^{(0)} + B_1^{(0)} \right) + \frac{1}{2} Z_{\frac{1}{2}} = \frac{1}{2} Z_1 + \frac{1}{2} Z_{\frac{1}{2}}.
\]
Then, define \(B_t^{(1)}\) by linear interpolation:
\[
B_t^{(1)} = (1-2t)B_0^{(1)} + 2tB_{\frac{1}{2}}^{(1)} = 2tB_{\frac{1}{2}}^{(1)}, \quad t \in [0, \frac{1}{2}],
\]
\[
B_t^{(1)} = (2-2t)B_{\frac{1}{2}}^{(1)} + (2t-1)B_1^{(1)} \quad t \in [\frac{1}{2}, 1].
\]
Note that

Then, interpolate linearly to build $B_{t}^{(n+1)}$ for all $t \in [0,1]$.  

**Lemma 1.4.** For all $k \in \{0, \ldots, 2^n - 1\}$, $B_{k+1}^{(n+1)} - B_{k}^{(n)}$ is independent of $B_{k+1}^{(n)} - B_{k}^{(n+1)}$, and

$$B_{k+1}^{(n+1)} - B_{k}^{(n)} \sim \mathcal{N}(0, \frac{1}{2^{n+1}}).$$

**Proof.** By induction. For $n = 0$, let us first check that $B_{\frac{1}{2}}^{(1)} - B_{0}^{(0)}$ is independent of $B_{\frac{1}{2}}^{(1)} - B_{\frac{1}{2}}^{(1)}$. Note that

$$B_{\frac{1}{2}}^{(1)} - B_{0}^{(0)} = \frac{1}{2} Z_{1} + \frac{1}{2} Z_{\frac{1}{2}} = \frac{1}{\sqrt{2}} \sqrt{Z_{1} + Z_{\frac{1}{2}}}.$$

$$B_{1}^{(1)} - B_{\frac{1}{2}}^{(1)} = \frac{1}{2} Z_{1} - \frac{1}{2} Z_{\frac{1}{2}} = \frac{1}{\sqrt{2}} \sqrt{Z_{1} - Z_{\frac{1}{2}}}.$$

Since $Z_{1}, Z_{\frac{1}{2}}$ are i.i.d. $\mathcal{N}(0,1)$, the Main Lemma (c.f. beginning of the proof) tells us that $B_{\frac{1}{2}}^{(1)} - B_{0}^{(0)}$ is independent of $B_{1}^{(1)} - B_{\frac{1}{2}}^{(1)}$ and that $B_{\frac{1}{2}}^{(1)} - B_{\frac{1}{2}}^{(1)}$ is $\mathcal{N}(0, \frac{1}{2}).$

Now, let $n \geq 1$, and assume that the property holds for $n - 1$. We have

$$B_{\frac{k+1}{2}}^{(n+1)} - B_{\frac{k}{2}}^{(n)} = \frac{1}{2} \left( B_{\frac{k}{2}}^{(n)} + B_{\frac{k+1}{2}}^{(n)} \right) + \frac{1}{2^{n+1}} \sqrt{2Z_{k+1}^{(n+1)} - B_{\frac{k+1}{2}}^{(n)}} = \frac{1}{2} B_{\frac{k+1}{2}}^{(n)} - \frac{1}{2} B_{\frac{k}{2}}^{(n)} + \frac{1}{2} \frac{1}{\sqrt{2^{2n}}} \left[ \sqrt{2^{2n}} \left( B_{\frac{k+1}{2}}^{(n)} - B_{\frac{k}{2}}^{(n)} \right) + Z_{2^{k+1}+1}^{(n)} \right].$$

By induction, $\left( B_{\frac{k}{2}}^{(n)} - B_{\frac{k}{2}}^{(n)} \right) \sim \mathcal{N}(0, \frac{1}{2^{n+1}})$. Also, $\left( B_{\frac{k+1}{2}}^{(n)} - B_{\frac{k}{2}}^{(n)} \right)$ and $Z_{2^{k+1}+1}^{(n)}$ are independent (since the $Z_{q}$’s are independent). Thus, by the Main Lemma again,

$$\frac{1}{\sqrt{2}} \left[ \sqrt{2^{2n}} \left( B_{\frac{k+1}{2}}^{(n)} - B_{\frac{k}{2}}^{(n)} \right) + Z_{2^{k+1}+1}^{(n)} \right]$$

is standard Gaussian. It follows that $B_{\frac{k+1}{2}}^{(n+1)} - B_{\frac{k}{2}}^{(n)} \sim \mathcal{N}(0, \frac{1}{2^{n+1}})$. Similarly, noting that

$$B_{\frac{k}{2}}^{(n)} - B_{\frac{k+1}{2}}^{(n+1)} = \frac{1}{\sqrt{2^{n+1}} \sqrt{2}} \left[ \sqrt{2^{2n}} \left( B_{\frac{k}{2}}^{(n)} - B_{\frac{k+1}{2}}^{(n)} \right) - Z_{2^{k+1}+1}^{(n)} \right],$$

we deduce the result by the Main Lemma again.

**Lemma 1.5.** For all $n \geq 0$, for all $p < q \in D_{n}$,

1. $B_{q}^{(n)} - B_{p}^{(n)} \sim \mathcal{N}(0, q - p)$.
2. $B_{q}^{(n)} - B_{p}^{(n)}$ is independent of $B_{r}^{(n)}$, for all $r \leq p$, $r \in D_{n}$.
Proof. This is a consequence of Lemma 1.4.

1. Let $p, q \in \mathcal{D}_n$. Then there exists $k < l$ such that $p = \frac{k}{2^n}$ and $q = \frac{l}{2^n}$. Hence,

$$B_p^{(n)} - B_p^{(n)} = B_{p \frac{k}{2^n}}^{(n)} - B_{q \frac{l}{2^n}}^{(n)} = B_{p \frac{k}{2^n}}^{(n)} - B_{q \frac{l}{2^n}}^{(n)} + \cdots + B_{q \frac{l}{2^n}}^{(n)} - B_{q \frac{l}{2^n}}^{(n)}.$$

One can see that each term of sum are mutually independent (proof similar to Lemma 1.4). By Lemma 1.4 each term is a Gaussian $\mathcal{N}(0, \frac{1}{2^n})$, hence $B_p^{(n)} - B_p^{(n)} \sim \mathcal{N}(0, q - p)$.

2. Same argument.

Lemma 1.5 holds for all $p < q \in \mathcal{D}$.

Proof. If $p, q \in \mathcal{D}$, then there exists $n \in \mathbb{N}$ such that $p, q \in \mathcal{D}_n$. Apply then Lemma 1.5.

Step 2: [Almost sure uniform convergence]

Let us denote, for $\omega \in \Omega$,

$$\Delta^{(n)}(\omega) = \max_{t \in [0,1]} |B^{(n+1)}(\omega) - B^{(n)}(\omega)| = \max_{k \in \{0, \ldots, 2^n - 1\}} \max_{t \in [\frac{k}{2^n}, \frac{k + 1}{2^n}]} |B^{(n+1)}(\omega) - B^{(n)}(\omega)|.$$

We drop the variable $\omega$ next. Since, by definition, $B^{(n)}_t$ is defined by linear interpolation and $B^{(n+1)}_t = B^{(n)}_t$ when $t \in \mathcal{D}_n$, we see that

$$\max_{t \in [\frac{k}{2^n}, \frac{k + 1}{2^n}]} |B^{(n+1)}_t - B^{(n)}_t|$$

is attained at the midpoint $t = \frac{2k + 1}{2^{n+1}}$ (draw a picture). Hence,

$$\Delta^{(n)} = \max_{k \in \{0, \ldots, 2^n - 1\}} |B^{(n+1)}_k - B^{(n)}_k| = \max_{k \in \{0, \ldots, 2^n - 1\}} |B^{(n+1)}_k - B^{(n)}_k| - \frac{1}{2} \left(B^{(n)}_k + B^{(n)}_{k+1}\right)|$$

$$= \frac{1}{2} \max_{k \in \{0, \ldots, 2^n - 1\}} \left|B^{(n+1)}_k - B^{(n)}_k\right| - \left(B^{(n)}_k - B^{(n+1)}_{k+1}\right)|.$$

Note that $B^{(n+1)}_k - B^{(n)}_k$ and $B^{(n)}_{k+1} - B^{(n+1)}_{k+1}$ are i.i.d. Gaussian $\mathcal{N}(0, \frac{1}{2^{2n+1}})$. Hence, for all $k$,

$$W^{(n)}_k = \left(B^{(n+1)}_k - B^{(n)}_k\right) - \left(B^{(n)}_{k+1} - B^{(n+1)}_{k+1}\right)$$

is Gaussian $\mathcal{N}(0, \frac{1}{2^{2n+1}})$. Let $\alpha \geq 1$. One has,

$$\mathbb{P}(\Delta^{(n)} \geq \frac{\alpha}{2\sqrt{2^n}}) = \mathbb{P}\left(\frac{1}{2} \max_{k \in \{0, \ldots, 2^n - 1\}} |W^{(n)}_k| \geq \frac{\alpha}{2\sqrt{2^n}}\right) \leq 2^n \mathbb{P}\left(\frac{1}{2} |W^{(n)}_0| \geq \frac{\alpha}{2\sqrt{2^n}}\right),$$

where the inequality comes from the union bound. Note that for $Z \sim \mathcal{N}(0, 1)$,

$$\mathbb{P}(Z \geq \alpha) \leq \frac{e^{-\frac{\alpha^2}{2}}}{\alpha \sqrt{2\pi}}.$$
hence, by symmetry of Gaussian and the fact that $\sqrt{2\pi}W^{(n)}_0 \sim \mathcal{N}(0,1)$,
\[
\mathbb{P}(\Delta^{(n)} \geq \frac{\alpha}{2\sqrt{2^n}}) = 2^{n+1}\mathbb{P}(\sqrt{2\pi}W^{(n)}_0 \geq \alpha) \leq 2^{n+1} \frac{e^{-\frac{\alpha^2}{2}}}{\alpha \sqrt{2\pi}}.
\]
Now, take $\alpha = 2\sqrt{n}$. Then,
\[
\mathbb{P}(\Delta^{(n)} \geq \frac{\sqrt{n}}{\sqrt{2^n}}) \leq \frac{1}{\sqrt{2\pi n}} \left(\frac{2}{e^2}\right)^n.
\]
Hence,
\[
\sum_{n \geq 1} \mathbb{P}(\Delta^{(n)} \geq \frac{\sqrt{n}}{\sqrt{2^n}}) < +\infty.
\]
By Borel-Cantelli,
\[
\mathbb{P}(\lim\sup\{\Delta^{(n)} \geq \frac{\sqrt{n}}{\sqrt{2^n}}\}) = 0.
\]
In other words, there exists $A \subset \Omega$, $\mathbb{P}(A) = 1$, such that for all $\omega \in \Omega$, there exists $N \in \mathbb{N}$, such that for all $n \geq N$,
\[
\Delta^{(n)}(\omega) \leq \frac{\sqrt{n}}{\sqrt{2^n}}.
\]
Recalling the definition of $\Delta^{(n)}(\omega)$, we thus proved that for all $\omega$ in a set $A$ of probability 1,
\[
\sum_{n \geq 1} \|B^{n+1}(\omega) - B^n(\omega)\|_{L^\infty([0,1])} < +\infty.
\]
A standard result of analysis allows us to conclude that, almost surely, $\{B^{(n)}(\omega)\}_{n \geq 1}$ converges uniformly on $[0,1]$. We then define
\[
B(\omega) = \left\{ \begin{array}{ll}
\lim_{n \to +\infty} B^{(n)}(\omega) & \text{if } \omega \in A \\
0 & \text{if } \omega \notin A.
\end{array} \right.
\]

Step 3: [The limit is a Brownian motion on $[0,1]$]

- **Continuity**: By construction, for all $\omega \in \Omega$, for all $n \in \mathbb{N}$, $t \mapsto B^{(n)}_t(\omega)$ is continuous on $[0,1]$. Since, almost surely, $\{B_n\}$ converges uniformly on $[0,1]$ to $B$, we deduce that, almost surely, $t \mapsto B_t(\omega)$ is continuous.

  - Since for all $n \in \mathbb{N}$, $B_0^{(n)} = 0$, we deduce that $B_0 = 0$.

- **Stationary increments**: Let $t, s \in D$. Then, there exists $m \in \mathbb{N}$ such that $t, s \in D_m$. Hence, $B^{(m)}_t - B^{(m)}_s \sim \mathcal{N}(0, t-s)$. By construction, for all $t \in D_m$, for all $n \geq m$, $B^{(n)}_t = B^{(m)}_t$. Hence
\[
B_t - B_s = \lim_{n \to +\infty} B^{(n)}_t - B^{(n)}_s = \lim_{n \to +\infty} B^{(m)}_t - B^{(m)}_s = B^{(m)}_t - B^{(m)}_s,
\]
where the limit is understood as “almost sure convergence”. Since $B^{(m)}_t - B^{(m)}_s \sim \mathcal{N}(0, t-s)$, we have $B_t - B_s \sim \mathcal{N}(0, t-s)$. Now, assume that $t, s \in [0,1]$. By density of $D$ in $[0,1]$, there exist sequences $\{t_k\}, \{s_k\} \in D$ such that $t = \lim_k t_k$ and $s = \lim_k s_k$. Since, almost surely, $t \mapsto B_t$ is continuous, we have, almost surely, $B_t = \lim_k B_{t_k}$ and $B_s = \lim_k B_{s_k}$. Since, for all $k$, $B_{t_k} - B_{s_k} \sim \mathcal{N}(0, t_k - s_k)$, we can conclude that $B_t - B_s \sim \mathcal{N}(0, t-s)$ (use, for example, characteristic functions).

- **Independent increments**: Same argument.
1.3 Simulation of Brownian motion

Fix an integer \( n \in \mathbb{N} \). Given times \( 0 = t_0 < t_1 < \cdots < t_n \), generate \( Z_1, \ldots, Z_n \) i.i.d. \( \mathcal{N}(0,1) \).

Define
\[
\begin{align*}
B_0 &= 0, \\
B_{t_1} &= \sqrt{t_1}Z_1, \\
B_{t_2} &= B_{t_1} + \sqrt{t_2-t_1}Z_2 = \sqrt{t_1}Z_1 + \sqrt{t_2-t_1}Z_2, \\
& \vdots \\
B_{t_n} &= B_{t_{n-1}} + \sqrt{t_n-t_{n-1}}Z_n = \sum_{i=1}^{n} \sqrt{t_i-t_{i-1}}Z_i
\end{align*}
\]

Using this construction, \( \{B_t\} \) is a Brownian motion at times \( 0 = t_0 < t_1 < \cdots < t_n \). Indeed, it starts at 0, and for all \( l \leq m < n \),
\[
B_{t_n} - B_{t_m} = \sum_{i=1}^{n} \sqrt{t_i-t_{i-1}}Z_i - \sum_{i=1}^{m} \sqrt{t_i-t_{i-1}}Z_i = \sum_{i=m+1}^{n} \sqrt{t_i-t_{i-1}}Z_i,
\]
which is Gaussian \( \mathcal{N}(0, t_n - t_m) \), and is independent of \( B_t \).

1.4 Properties of the Brownian motion

Definition 1.7. \( \{X_t\}_{t \geq 0} \) is a Gaussian process if for all \( n \in \mathbb{N} \), for all \( t_1 < \cdots < t_n \), the random vector \((X_{t_1}, \ldots, X_{t_n})\) is multivariate Gaussian.

Theorem 1.1. \((X_1, \ldots, X_n)\) is multivariate Gaussian \( \iff \) every linear combination of the \( X_i \)'s is Gaussian, that is, for all \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \), \( \lambda_1X_1 + \cdots + \lambda_nX_n \) is Gaussian \( \iff \)
\[ \exists \mu \in \mathbb{R}^n, \exists A \in \mathbb{R}^{n \times m}, (X_1, \ldots, X_n) = \mu + A(Z_1, \ldots, Z_n), \]
where \( Z_1, \ldots, Z_n \) are i.i.d. \( \mathcal{N}(0,1) \).

Theorem 1.2. A Brownian motion is a Gaussian process.

Proof. Define
\[
Z_j = \frac{B_{t_j} - B_{t_{j-1}}}{\sqrt{t_j-t_{j-1}}}, \quad j = 1, \ldots, n.
\]
In particular, the \( Z_j \)'s are i.i.d. standard Gaussian \( \mathcal{N}(0,1) \). Note that
\[
\begin{pmatrix}
B_{t_1} \\
\vdots \\
B_{t_n}
\end{pmatrix} =
\begin{pmatrix}
\sqrt{t_1} & 0 & \cdots & 0 \\
\sqrt{t_2-t_1} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
\sqrt{t_1} & \cdots & \sqrt{t_n-t_{n-1}}
\end{pmatrix}
\begin{pmatrix}
Z_1 \\
\vdots \\
Z_n
\end{pmatrix}.
\]
Hence \( \{B_t\} \) is a Gaussian process. \( \square \)

Definition 1.8. Let \( \{\mathcal{F}_t\} \) be a filtration. The germ \( \sigma \)-algebra is
\[
\mathcal{F}_s^+ = \cap_{t>s} \mathcal{F}_t.
\]

Remark 1.9. 1. In general \( \mathcal{F}_s^+ \neq \mathcal{F}_s \). Indeed, let \( X \) be a non-constant random variable. Define \( X_t = tX \), \( t \geq 0 \), and \( \mathcal{F}_t = \sigma(X_s : 0 \leq s \leq t) \). Note that for all \( t > 0 \), \( \mathcal{F}_t = \sigma(X) \). However,
\[
\{\emptyset, \Omega\} = \mathcal{F}_0 \neq \cap_{t>0} \mathcal{F}_t = \sigma(X).
\]
2. $\mathcal{F}_s^+$ represents an infinitesimal additional information into the future.

**Theorem 1.3** (Blumenthal 0-1 Law). Let $\{B_t\}$ be a Brownian motion. If $A \in \mathcal{F}_0^+$, then $\mathbb{P}(A) = 0$ or 1.

**Corollary 1.10.** Let $\{B_t\}$ be a standard Brownian motion. Define

$$T_1 = \inf\{t > 0 : B_t > 0\}, \quad T_2 = \inf\{t > 0 : B_t = 0\}, \quad T_3 = \inf\{t > 0 : B_t < 0\}.$$ 

Then, $\mathbb{P}(T_1 = 0) = \mathbb{P}(T_2 = 0) = \mathbb{P}(T_3 = 0) = 1$.

**Proof.** One has

$$\{T_1 = 0\} = \cap_{n \geq 1} \cup_{\varepsilon \in (0, \frac{1}{n})} \{B_s > 0\}.$$ 

Hence, $\{T_1 = 0\} \in \mathcal{F}_0^+$. Note that for all $t > 0$,

$$\{B_t > 0\} \subset \{T_1 \leq t\},$$

hence

$$\mathbb{P}(T_1 \leq t) \geq \mathbb{P}(B_t > 0) = \frac{1}{2}.$$ 

We deduce that

$$\mathbb{P}(T_1 = 0) = \mathbb{P}(\cap_n \{T_1 \leq \frac{1}{n}\}) = \lim_n \mathbb{P}(T_1 \leq \frac{1}{n}) \geq \frac{1}{2}.$$ 

Since $\{T_1 = 0\} \in \mathcal{F}_0^+$, by Blumenthal 0-1 law, we conclude that $\mathbb{P}(T_1 = 0) = 1$.

By symmetry, (that is, $\{-B_t\}$ is a Brownian motion), $\mathbb{P}(T_3 = 0) = 1$.

With probability 1, $t \mapsto B_t$ is continuous and satisfies $\mathbb{P}(T_1 = 0) = \mathbb{P}(T_3 = 0) = 1$, hence by the intermediate value theorem, $\mathbb{P}(T_2 = 0) = 1$. \hfill \Box

**Remark 1.11.** In particular, Corollary 1.10 tells us that with proba 1, for all $\varepsilon > 0$, $B_t$ hits 0 infinitely many times in the interval $(0, \varepsilon)$.

**Theorem 1.4** (Long term behavior of Brownian motion). Let $\{B_t\}$ be a Brownian motion, then

$$\limsup_{t \to +\infty} \frac{B_t}{\sqrt{t}} = +\infty \quad \text{and} \quad \liminf_{t \to +\infty} \frac{B_t}{\sqrt{t}} = -\infty.$$ 

**Proof.** Fix $M > 0$.

$$\mathbb{P}(\limsup_{t \to +\infty} \frac{B_t}{\sqrt{t}} > M) = \mathbb{P}(\limsup_{t \to +\infty} \sqrt{t}B_\frac{1}{t} > M) = \mathbb{P}(\cap_{t > 0} \cup_{0 \leq s \leq t} \{\sqrt{s}B_{\frac{1}{t}} > M\})$$

**Fact:** $\{sB_{\frac{1}{t}}\}$ is a Brownian motion (Time inversion — see later).

**Fact:** $\{\limsup f_t > M\} = \limsup\{f_t > M\}$.

Note that $\sqrt{s}B_{\frac{1}{t}} = X_s$, where $X_s = sB_{\frac{1}{t}}$ being a Brownian motion. Hence,

$$\cap_{t > 0} \cup_{0 \leq s \leq t} \{\sqrt{s}B_{\frac{1}{t}} > M\} = \cap_{t > 0} \cup_{0 \leq s \leq t} \{X_s > M\sqrt{s}\} \in \mathcal{F}_0^+.$$ 

By Blumenthal 0-1 law,

$$\mathbb{P}(\limsup_{t \to +\infty} \frac{B_t}{\sqrt{t}} > M) = 0 \quad \text{or} \quad 1.$$ 

Now, note that

$$\mathbb{P}(\limsup_{t \to +\infty} \frac{B_t}{\sqrt{t}} > M) \geq \mathbb{P}(\limsup_{n \to +\infty} \frac{B_n}{\sqrt{n}} > M) = \mathbb{P}(\cap_{n \geq 1} \cup_{k \geq n} \{\frac{B_k}{\sqrt{k}} > M\})$$

$$= \lim_{n \to +\infty} \mathbb{P}(\cup_{k \geq n} \{\frac{B_k}{\sqrt{k}} > M\}) \geq \lim_{n \to +\infty} \mathbb{P}(\{\frac{B_n}{\sqrt{n}} > M\}) = \lim_{n \to +\infty} \mathbb{P}(\{B_1 > M\}) = \mathbb{P}(B_1 > M) > 0.$$ 

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We conclude that
\[ P(\limsup_{t \to +\infty} \frac{B_t}{\sqrt{t}} > M) = 1. \]

It follows that
\[ P(\limsup_{t \to +\infty} \frac{B_t}{\sqrt{t}} = +\infty) = P(\cap_{M>0} \limsup_{t \to +\infty} \frac{B_t}{\sqrt{t}} > M) = \lim_{M \to +\infty} P(\limsup_{t \to +\infty} \frac{B_t}{\sqrt{t}} > M) = 1. \]

By symmetry \((-B_t)\) is a Brownian motion, we deduce that
\[ P(\liminf_{t \to +\infty} \frac{B_t}{\sqrt{t}} = -\infty) = 1. \]

**Remark 1.12.** In other words, a Brownian motion is recurrent (each value \(a \in \mathbb{R}\) is visited infinitely many often).

**Definition 1.13.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Recall that a filtration \(\{F_t\}_{t \geq 0}\) is a family of sigma-algebras such that for all \(s \leq t\), \(F_s \subset F_t \subset \mathcal{F}\).

A process \(\{M_t\}_{t \geq 0}\) is a \(\{F_t\}\) continuous-time martingale if

i) For all \(t \geq 0\), \(M_t\) is \(F_t\)-measurable.

ii) For all \(t \geq 0\), \(E[|M_t|] < +\infty\).

iii) For all \(s \leq t\), \(E[M_t | F_s] = M_s\).

**Proposition 1.14.** A Brownian motion is a continuous-time martingale.

**Proof.**
\[ E[B_t | F_s] = E[B_t - B_s + B_s | F_s] = E[B_t - B_s | F_s] + B_s = B_s, \]
because \(B_t - B_s\) is independent of \(F_s\) and has expectation 0.

**Theorem 1.5 (Law of Large Numbers for Brownian motion).** For a Brownian motion \(\{B_t\}\), \(\lim_{t \to +\infty} \frac{B_t}{\sqrt{t}} = 0\) almost surely.

**Proof.**
**Step 1:** Note that \(B_n = B_1 - B_0 + \cdots + B_n - B_{n-1}\), so we can write
\[ B_n = \sum_{k=1}^{n} X_k, \]
where \(X_k = B_k - B_{k-1}\). Note that \(\{X_k\}\) is a sequence of i.i.d. \(\mathcal{N}(0,1)\) random variable. Hence, by the strong LLN, \(\frac{B_n}{\sqrt{n}} \to E[B_1] = 0\) almost surely.

**Step 2:** We will prove that
\[ \sum_{n \geq 0} P(\sup_{t \in [n,n+1]} |B_t - B_n| > n^{3/2}) < +\infty. \]

Fix \(n \geq 0\). Define, for \(m \geq 0\) and \(k \in \{0, \ldots, 2^m\}\),
\[ X_k = B_{n + k \cdot 2^m} - B_n. \]

Since \(\{B_{n+t} - B_n\}_{t \geq 0}\) is a Brownian motion, it is a martingale. It follows that \(\{X_k\}\) is a discrete time martingale. We can thus apply Doob’s inequality and obtain
\[ P(\sup_{0 \leq k \leq 2^m} |X_k| > n^{3/2}) \leq \frac{E[|X_{2^m}|^2]}{n^{3/2}} = \frac{E[|B_{n+1} - B_n|^2]}{n^{3/2}} = \frac{1}{n^{3/2}}. \]
Because $t \mapsto B_t$ is continuous, we have
\[
\left\{ \sup_{t \in [n,n+1]} |B_t - B_n| > n^{\frac{2}{3}} \right\} = \bigcup_{m \geq 1} \left\{ \sup_{0 \leq k \leq 2^m} |X_k| > n^{\frac{2}{3}} \right\}.
\]
Hence,
\[
\mathbb{P}\left( \sup_{t \in [n,n+1]} |B_t - B_n| > n^{\frac{2}{3}} \right) = \lim_{m \to +\infty} \mathbb{P}\left( \sup_{0 \leq k \leq 2^m} |X_k| > n^{\frac{2}{3}} \right) \leq \frac{1}{n^{\frac{2}{3}}}.
\]

**Step 3:** Define, for $n \geq 0$,
\[
A_n = \left\{ \sup_{t \in [n,n+1]} |B_t - B_n| > n^{\frac{2}{3}} \right\}.
\]
Since $\sum \mathbb{P}(A_n) < +\infty$, by Borel-Cantelli we have $\mathbb{P}(\lim \sup A_n) = 0$. This means that, for all $\omega$ in a set of probability 1,
\[
\exists n_0 \geq 1, \forall n \geq n_0, \forall t \in [n,n+1], \left| \frac{B_t(\omega)}{t} \right| \leq n \left( \frac{|B_t(\omega) - B_{n}(\omega)|}{n} + \frac{|B_n(\omega)|}{n} \right) \leq \frac{1}{n^{\frac{2}{3}}} + \frac{|B_n(\omega)|}{n},
\]
which goes to 0 as $n \to +\infty$.

**Corollary 1.15 (Time Inversion).** Let $\{B_t\}$ be a Brownian motion. The process $\{X_t\}_{t \geq 0}$ defined by $X_t = tB_{\frac{1}{t}}$ for $t > 0$ and $X_0 = 0$, is a Brownian motion, for the natural filtration $\mathcal{F}_t = \sigma(X_s : s \leq t)$.

**Proof.** **Continuity at 0:** From Theorem 1.5, we have
\[
\lim_{t \to 0^+} X_t = \lim_{t \to +\infty} X_{\frac{1}{t}} = \lim_{t \to +\infty} \frac{B_t}{t} = 0 = X_0.
\]
**Gaussian Increments:** Note that, for $s \leq t$,
\[
X_t - X_s = (t - s)\frac{B_t}{t} - s(\frac{B_1}{t} - B_{\frac{1}{t}}),
\]
which is $\mathcal{N}(0,t-s)$.

**Independent Increments:** Since $(X_t - X_s, X_s)$ is a bivariate Gaussian, we can conclude independence because $\mathbb{E}[(X_t - X_s)X_s] = 0$. 

### 1.5 Reflection Principle

**Definition 1.16.** Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered space. Let $\{X_t\}$ be a stochastic process adapted to $\{\mathcal{F}_t\}$. We say that $\{X_t\}$ is a Markov process if
\[
\forall A \in \mathcal{F}, \forall h \geq 0, \forall t \geq 0, \quad \mathbb{P}(X_{t+h} \in A|\mathcal{F}_t) = \mathbb{P}(X_{t+h} \in A|X_t).
\]

**Notation:**
\[
\mathbb{P}(X_{t+h} \in A|X_t) = \mathbb{P}(X_{t+h} \in A|\sigma(X_t)) = \mathbb{E}[1_A(X_{t+h})|\sigma(X_t)].
\]

**Theorem 1.6.** A Brownian motion is a Markov process (w.r.t the same filtration).

**Sketch of proof.** We want to prove that
\[
\mathbb{P}(B_{t+h} \in A|\mathcal{F}_t) = \mathbb{P}(B_{t+h} \in A|B_t),
\]
equivalently,
\[
\mathbb{E}[1_A(B_{t+h})|\mathcal{F}_t] = \mathbb{E}[1_A(B_{t+h})|\sigma(B_t)].
\]
Let $\Phi : \mathbb{R} \to \mathbb{R}$ be measurable, then
\[
\mathbb{E}[\Phi(B_{t+h})|\mathcal{F}_t] = \mathbb{E}[\Phi(B_{t+h} - B_t + B_t)|\mathcal{F}_t] = \mathbb{E}[g(X, B_t)|\mathcal{F}_t],
\]
where $X = B_{t+h} - B_t$, which is independent of $\mathcal{F}_t$, and $g(x,y) = \Phi(x+y)$. Since $X$ is independent of $\mathcal{F}_t$, and $B_t$ is $\sigma(B_t)$-measurable, $\mathbb{E}[g(X, B_t)|\mathcal{F}_t] = \mathbb{E}[g(X, B_t)|\sigma(B_t)]$. To prove this, start with functions $g$ of the form $g(x,y) = 1_C(x)1_D(y)$, and use the fact that they approximate any Borel function. 

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**Definition 1.17.** A random variable $T$ is an $\{F_t\}$-stopping time if
\[ \forall t \geq 0, \quad \{T \leq t\} \in F_t. \]

**Proposition 1.18.** 1. Every deterministic time is a stopping time.
2. If $\{T_n\}$ is a sequence of stopping time, the $\sup_n T_n$ is a stopping time.

*Proof.* 1. Exercise.
2. Fix $t \geq 0$. Then,
\[ \{\sup_n T_n \leq t\} = \cap_n \{T_n \leq t\} \in F_t. \]

**Remark 1.19.** In general, $\inf_n T_n$ is not a stopping time. Indeed, recalling that if $m = \inf(A)$, then for all $\varepsilon > 0$, there exists $a \in A$, such that $m \geq a - \varepsilon$. In particular we have
\[ \{\inf_n T_n \leq t\} = \cap_{\varepsilon>0} \cup_{n \geq 1} \{T_n \leq t + \varepsilon\} \in \cap_{\varepsilon>0} F_{t+\varepsilon} = F_{t+}. \]
Since in general $F_t \neq F_{t+}$, it follows that $\inf_n T_n$ is not a stopping time.

Similarly, note that, when $F_t = \sigma(B_s : s \leq t)$,
1. If $F$ is a closed set, then $T = \inf\{t \geq 0 : B_t \in F\}$ is a stopping time.
2. If $O$ is open, then $T = \inf\{t \geq 0 : B_t \in O\}$ is not a stopping time.

**Definition 1.20.** A filtration $\{F_t\}$ is right-continuous if for all $t \geq 0$, $F_t = F_{t+}$.

**Example 1.21.** The canonical filtration for a Brownian motion $\{B_t\}$:
Define
\[ F_t = \sigma(B_s : s \leq t), \quad t \geq 0, \]
and
\[ \widetilde{F}_t = F_{t+} = \cap_{\varepsilon>0} F_{t+\varepsilon}, \quad t \geq 0. \]
Then $\{\widetilde{F}_t\}$ is a right-continuous filtration, and $\{B_t\}$ is adapted to $\{\widetilde{F}_t\}$.

**Proposition 1.22.** 1. If $\{T_n\}$ is a sequence of $\{F_t^+\}$-stopping times, then $\inf_n T_n$ is an $\{F_t^+\}$-stopping time.
2. If $O$ is open, then $T = \inf\{t \geq 0 : B_t \in O\}$ is a $\{\widetilde{F}_t\}$-stopping time.

**Definition 1.23.** For a stopping time $T$, define
\[ F_T = \{A \in F : A \cap \{T \leq t\} \in F_{t+}, \forall t \geq 0\}. \]

**Theorem 1.7.** $F_T$ is a $\sigma$-algebra.

*Proof.* Same proof as in the discrete case. \qed

**Definition 1.24.** Let $(\Omega, F, \{F_n\}, P)$ be a filtered space. Let $\{X_t\}$ be a stochastic process adapted to $\{F_t\}$. We say that $\{X_t\}$ is a strong Markov process if for all stopping time $T$ finite almost surely,
\[ \forall A \in F, \forall h \geq 0, \quad P(X_{T+h} \in A|F_T) = P(X_{T+h} \in A|X_T). \]

**Theorem 1.8.** The Brownian motion is a strong Markov process.

*Sketch of Proof.* Note that $\{B_{T+t} - B_T\}_{t \geq 0}$ is a standard Brownian motion independent of $F_T$. \qed
Theorem 1.9 (Reflection principle). Let $T$ be a stopping time and $\{B_t\}$ be a standard Brownian motion.

If $M = (x, y)$, then the reflection of $M$ with respect to the line passing through $(0, a)$ and parallel to the $x$-axis is $M^* = (x, 2a - y)$ (draw a picture).

For $t \geq 0$, define

$$B_t^* = B_t1_{t \leq T} + (2B_T - B_t)1_{t > T}.$$ 

Then, $\{B_t^*\}$ is a standard Brownian motion.

Definition 1.25. The process $B_t^*$ defined in Theorem 1.9 is called reflected Brownian motion.

Corollary 1.26. Let $\{B_t\}$ be a Brownian motion. Consider, for $t \geq 0$,

$$M_t = \sup_{0 \leq s \leq t} B_s.$$ 

Then, $M_t \sim |Z|$, where $Z \sim N(0, t)$. This means that supremum of Brownian motion path has a $\chi$ distribution.

Proof. First note that $P(M_t \geq 0) = 1$ because $B_0 = 0$ a.s.

Fix $a > 0$. Let us find $P(M_t \geq a)$. Consider $\{B_t^*\}$ the reflected Brownian motion with respect to the stopping time $T_\alpha = \inf\{t \geq 0 : B_t = a\}$. Note that

i) $\{B_t \geq a\} \subseteq \{M_t \geq a\}$.

ii) $\{M_t \geq a\} \cap \{B_t < a\} = \{B_t^* > a\}$.

The point i) is clear. The inclusion $\subseteq$ of the point ii) is clear from the picture (after reflection, $B_t < a$ if and only if $B_t^* > a$). For the other inclusion $\supseteq$, if $\{B_t^* > a\}$, then either $\{B_t > a\}$ either $\{B_t < a\}$. The case $\{B_t > a\}$ is impossible because $B_t > a$ implies that $T_\alpha < t$. Necessarily, $\{B_t < a\}$. Since $\{B_t < a\}$ and $\{B_t^* > a\}$, we have $T_\alpha \leq t$ and thus $M_t \geq a$.

Thus, from ii) and Theorem 1.9,

$$P(M_t \geq a, B_t < a) = P(B_t^* > a) = P(B_t > a).$$

Hence,

$$P(M_t \geq a) = P(M_t \geq a, B_t < a) + P(M_t \geq a, B_t \geq a) = 2P(B_t \geq a) = P(|B_t| \geq a).$$

\[\square\]

1.6 Differentiability of the paths of the Brownian motion

Theorem 1.10. With probability 1, the paths of the Brownian motion are nowhere differentiable. Formally, let $\{B_t\}$ be a Brownian motion, then

$$P\left\{\omega \in \Omega : \exists t_0 \in [0, +\infty), t \mapsto B_t(\omega) \text{ is differentiable at } t_0\right\} = 0.$$ 

Proof. Step 1: [Setup]

Without loss of generality, let us prove the result on $[0, 1]$. Denote

$$A = \{\omega \in \Omega : \exists t_0 \in [0, 1], t \mapsto B_t(\omega) \text{ is differentiable at } t_0\}.$$ 

We want to prove that $P(A) = 0$. For $n \geq 3$ and $k \in \{1, \ldots, n - 2\}$, define

$$M_{k,n} = \max\{|B_{k+2} - B_{k+1}n|, |B_{k+1} - B_{k}n|, |B_{k} - B_{k-1}n|\},$$

and

$$M_n = \min(M_{1,n}, \ldots, M_{n-2,n}).$$
Step 2: The goal of Step 2 is to prove that

\[ \forall \omega \in A, \exists M \in \mathbb{N}, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, M_n(\omega) \leq \frac{5}{n}(1 + M). \]

Let \( \omega \in A \) (we drop the dependence on \( \omega \) next). Then there exists \( t_0 \in [0, 1] \) such that \( t \mapsto B_t \) is differentiable at \( t_0 \). By definition of differentiability, there exists \( L \in \mathbb{R} \) and \( \delta > 0 \) such that for all \( t \in [0, 1] \setminus \{t_0\} \), if \( |t - t_0| \leq \delta \), then \( |B_t - B_{t_0} - L(t - t_0)| \leq |t - t_0| \) (taking \( \varepsilon = 1 \)). Hence, by triangular inequality, for all \( t \) such that \( |t - t_0| \leq \delta \),

\[ |B_t - B_{t_0}| \leq (1 + |L|)|t - t_0|. \]

Now, note that there exists \( n_0 \geq 1 \) and \( k \in \{1, \ldots, n_0\} \), such that

\[ t_0 \in \left[ \frac{k-1}{n_0}, \frac{k}{n_0} \right] \quad \text{and} \quad \left| \frac{k+2}{n_0} - \frac{k-1}{n_0} \right| = \frac{3}{n_0} \leq \delta. \]

Let \( n \geq n_0 \). Then there exists \( k \in \{1, \ldots, n\} \) such that the above holds. Hence,

\[ |B_{\frac{k}{n}} - B_{\frac{k+1}{n}}| \leq |B_{\frac{k}{n}} - B_{t_0}| + |B_{t_0} - B_{\frac{k+1}{n}}| \leq (1 + |L|) \left( \left| \frac{k}{n} - t_0 \right| + \left| t_0 - \frac{k-1}{n} \right| \right) \leq \frac{2}{n}(1 + |L|). \]

Similarly, we have

\[ |B_{\frac{k+1}{n}} - B_{\frac{k}{n}}| \leq \frac{3}{n}(1 + |L|) \quad \text{and} \quad |B_{\frac{k+2}{n}} - B_{\frac{k+1}{n}}| \leq \frac{5}{n}(1 + |L|). \]

We thus proved that for all \( n \geq n_0 \), there exists \( k \in \{1, \ldots, n\} \) such that

\[ M_{k,n} \leq \frac{5}{n}(1 + |L|). \]

By definition of \( M_n \), this tells us that for all \( n \geq n_0 \),

\[ M_n \leq \frac{5}{n}(1 + |L|). \]

Now, just take any integer \( M \) greater than \( |L| \) to conclude that

\[ \forall \omega \in A, \exists M \in \mathbb{N}, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, M_n(\omega) \leq \frac{5}{n}(1 + M). \]

Equivalently,

\[ A \subset \bigcup_{M \in \mathbb{N}} \bigcup_{n_0 \in \mathbb{N}} \bigcap_{n \geq n_0} \left\{ M_n \leq \frac{5}{n}(1 + M) \right\}. \]

Step 3: The goal of Step 3 is to prove that

\[ \forall M \in \mathbb{N}, \lim_{n \to +\infty} \mathbb{P}(M_n \leq \frac{5}{n}(1 + M)) = 0. \]

Let \( n \geq 3 \) and \( k \in \{1, \ldots, n-2\} \). Denote,

\[ X_1 = |B_{\frac{k}{n}} - B_{\frac{k+1}{n}}|, \quad X_2 = |B_{\frac{k+1}{n}} - B_{\frac{k+2}{n}}|, \quad X_3 = |B_{\frac{k+2}{n}} - B_{\frac{k+3}{n}}|. \]

Since \( \{B_t\} \) is a Brownian motion, \( X_1, X_2, X_3 \) are i.i.d. with same distribution as \( |Z| \) where \( Z \sim \mathcal{N}(0, \frac{1}{n}) \). Thus, the CDF of \( M_{k,n} = \max(X_1, X_2, X_3) \) is

\[ F_{M_{k,n}}(x) = \mathbb{P}(M_{k,n} \leq x) = \mathbb{P}(X \leq x)^3, \quad x \in \mathbb{R}. \]
Note that
\[ P(X_1 \leq x) = P(|Z| \leq x\sqrt{n}), \]
where \( Z \sim \mathcal{N}(0, 1) \). Hence,
\[ P(X_1 \leq x) = \frac{2}{\sqrt{2\pi}} \int_0^{x\sqrt{n}} e^{-t^2/2} dt \leq \frac{2x\sqrt{n}}{\sqrt{2\pi}}. \]
We deduce that for all \( M \in \mathbb{N} \),
\[ P(M_{k,n} \leq \frac{5}{n}(1 + M)) \leq \left[ \frac{10}{\sqrt{2\pi}} (1 + M) \frac{1}{\sqrt{n}} \right]^3 \leq C n^{3/2}, \]
where \( C = \left[ \frac{10}{\sqrt{2\pi}} (1 + M) \right]^3 \). Hence, by union bound,
\[ P \left( M_n \leq \frac{5}{n}(1 + M) \right) = P \left( \bigcup_{k=1}^{n-2} \left\{ M_{k,n} \leq \frac{5}{n}(1 + M) \right\} \right) \leq \sum_{k=1}^{n-2} P \left( M_{k,n} \leq \frac{5}{n}(1 + M) \right) \leq \frac{C}{\sqrt{n}}. \]
We conclude that
\[ \forall M \in \mathbb{N}, \lim_{n \to +\infty} P(M_n \leq \frac{5}{n}(1 + M)) = 0. \]

**Step 4: [Conclusion]**
From Step 2,
\[ P(A) \leq P \left( \bigcup_{M \in \mathbb{N}} \bigcup_{n_0 \in \mathbb{N}} \bigcap_{n \geq n_0} \left\{ M_n \leq \frac{5}{n}(1 + M) \right\} \right). \]
Denote \( B_{n_0} = \bigcap_{n \geq n_0} \left\{ M_n \leq \frac{5}{n}(1 + M) \right\} \), and note that \( \{B_{n_0}\} \) is an increasing sequence of sets, hence, from Step 3,
\[ P \left( \bigcup_{n_0 \in \mathbb{N}} \bigcap_{n \geq n_0} \left\{ M_n \leq \frac{5}{n}(1 + M) \right\} \right) = \lim_{n_0 \to +\infty} P \left( \bigcap_{n \geq n_0} \left\{ M_n \leq \frac{5}{n}(1 + M) \right\} \right) \leq \lim_{n_0 \to +\infty} P \left( \left\{ M_{n_0} \leq \frac{5}{n_0}(1 + M) \right\} \right) \leq 0. \]
We conclude that
\[ P(A) \leq \sum_{M \in \mathbb{N}} P \left( \bigcup_{n_0 \in \mathbb{N}} \bigcap_{n \geq n_0} \left\{ M_n \leq \frac{5}{n}(1 + M) \right\} \right) = 0. \]
\[ \square \]