University of Florida

Lecture Notes - Brownian Motion

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1 Brownian motion

Recall that a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ is given.

1.1 Definitions

Definition 1.1 (Standard Brownian motion)**.** A continuous-time stochastic process $\{B_t\}_{t\in[0,+\infty)}$ is a standard Brownian motion if

1. $B_0 = 0$ a.s.
2. (Stationary Gaussian increments) $\forall\, 0 \leq s < t$, $B_t - B_s \sim B_{t-s} - B_0$ and $B_t - B_s \sim \mathcal{N}(0, t-s)$ (Gaussian of mean 0 and variance $t-s$).
3. (Independent increments) $\forall\, 0 \leq s < t$, $B_t - B_s$ is independent of $\mathcal{F}_s$.
4. With probability 1, the trajectories are continuous. Precisely:

$$\exists\, A \subset \Omega,\ \mathbb{P}(A) = 1, \forall\, \omega \in \Omega,\ t \mapsto B_t(\omega) \text{ is continuous on } [0, +\infty).$$

Remark 1.2. One may ask whether all the assumptions are necessary in the definition of the Brownian motion. Or, in other words, does one or several assumptions imply another one.

• The continuity assumption is a necessity. To see this, let $\{B_t\}$ be a Brownian motion and let U be uniformly distributed on $[0, 1]$. Define, for $\omega \in \Omega$ and $t \geq 0$,

$$\widetilde{B}_t(\omega) = B_t(\omega)1_{\{t\neq U(\omega)\}} + (1 + B_t(\omega))1_{\{t=U(\omega)\}}.$$

In this case, for all $t \geq 0$, $\mathbb{P}(\widetilde{B}_t = B_t) = 1$, and hence $\widetilde{B}_t$ satisfies properties 1-3. of the definition. However, for all $\omega \in \Omega$, $t \mapsto \widetilde{B}_t(\omega)$ is discontinuous (at $t = U(\omega)$).

• It can be shown that if 3-4 and stationary increments hold, then necessarily the increments are Gaussian.

• Property 1. is just a normalization. A brownian motion can start at any point.

• We will always consider the natural filtration $\mathcal{F}_t = \sigma(B_s : s \leq t)$.

Model: Brownian motions are used to model the trajectories of small particles in a fluid, or the evolution of the stock market. Generally speaking, it is used to model erratic motions.

Remark 1.3. When we say "Let $\{B_t\}_{t\geq 0}$ be a Brownian motion", we implicitly assume the existence of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a family of random variables $\{B_t\}$ on $(\Omega, \mathcal{F})$ such that $\mathbb{P}$ makes $\{B_t\}$ a Brownian motion (that is, such that $\{B_t\}$ satisfies the definitions 1-4. with respect to $\mathbb{P}$).

Question: Does such a probability space exist?

Answer: Yes, but technical to prove. This is the goal of the next section.

1.2 Construction of the Brownian motion

We will restrict the construction to $[0,1]$. For $n \geq 0$, denote

$$\mathcal{D}_n = \left\{ \frac{k}{2^n} : k \in \{0, \dots, 2^n\} \right\}.$$

For example,

$$D_0 = \{0,1\}, \quad D_1 = \left\{0, \frac{1}{2}, 1\right\}, \quad \mathcal{D}_2 = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}.$$

Denote

$$\mathcal{D} = \cup_{n \geq 0} D_n,$$

the dyadic of $[0,1]$. Before starting, first note that $\mathcal{D}$ is dense in $[0,1]$, and that $\{\mathcal{D}_n\}$ is increasing ($\mathcal{D}_n \subset \mathcal{D}_{n+1}$).

The process will follow the following steps:

Step 1: For each $n \in \mathbb{N}$, build a continuous process $\{B_t^{(n)}\}_{t \in [0,1]}$ that satisfies the properties of the Brownian motion on $\mathcal{D}_n$.

Step 2: With probability 1, $t \mapsto B_t^{(n)}$ converges uniformly on $[0,1]$.

Step 3: $\lim_{n \to +\infty} B_t^{(n)}$ is a Brownian motion.

Step 1: [Construction on the dyadic]

Let $\{Z_q\}_{q \in \mathcal{D}}$ be a sequence of i.i.d. standard Gaussian. In particular, for all $q \neq r \in \mathcal{D}$, Z_q is independent of Z_r, and $Z_q \sim \mathcal{N}(0,1)$.

Main Lemma: If X, Y are i.i.d. $\mathcal{N}(0,1)$, then $\frac{X+Y}{\sqrt{2}}$ and $\frac{X-Y}{\sqrt{2}}$ are i.i.d. $\mathcal{N}(0,1)$.

Proof: Exercise.

For each $\omega \in \Omega$, we are going to build $B_t^{(n)}(\omega)$ by induction on $n \in \mathbb{N}$, for $t \in \mathcal{D}_n$, and then interpolate linearly. We drop the variable ω next.

For $n = 0$:
Set $B_0^{(0)} = 0$ and $B_1^{(0)} = Z_1$. Then, we interpolate linearly between $B_0^{(0)}$ and $B_1^{(0)}$:

$$B_t^{(0)} = (1-t) B_0^{(0)} + t B_1^{(0)} = t Z_1, \quad t \in [0,1].$$

For $n = 1$:
Set

$$B_0^{(1)} = B_0^{(0)} = 0, \quad B_1^{(1)} = B_1^{(0)} = Z_1, \quad B_{\frac{1}{2}}^{(1)} = \frac{1}{2}\left(B_0^{(0)} + B_1^{(0)}\right) + \frac{1}{2} Z_{\frac{1}{2}} = \frac{1}{2} Z_1 + \frac{1}{2} Z_{\frac{1}{2}}.$$

Then, define $B_t^{(1)}$ by linear interpolation:

$$B_t^{(1)} = (1-2t) B_0^{(1)} + 2t B_{\frac{1}{2}}^{(1)} = 2t B_{\frac{1}{2}}^{(1)}, \quad t \in [0, \frac{1}{2}],$$

$$B_t^{(1)} = (2-2t) B_{\frac{1}{2}}^{(1)} + (2t-1) B_1^{(1)} \quad t \in [\frac{1}{2}, 1].$$

We continue this process for each $n \geq 0$.

For $n+1$:
Let $n \geq 0$. Assume $B_t^{(n)}$ built. For $k \in \{0, \dots, 2^n - 1\}$, define

$$B^{(n+1)}_{\frac{2k+1}{2^{n+1}}} = \frac{1}{2}\left(B^{(n)}_{\frac{k}{2^n}} + B^{(n)}_{\frac{k+1}{2^n}}\right) + \frac{1}{2^{\frac{n+2}{2}}} Z_{\frac{2k+1}{2^{n+1}}},$$

and for $t \in D_n$, define

$$B_t^{(n+1)} = B_t^{(n)}.$$

Then, interpolate linearly to build $B_t^{(n+1)}$ for all $t \in [0,1]$.

Lemma 1.4. For all $k \in \{0, \dots, 2^n - 1\}$, $B^{(n+1)}_{\frac{2k+1}{2^{n+1}}} - B^{(n)}_{\frac{k}{2^n}}$ is independent of $B^{(n)}_{\frac{k+1}{2^n}} - B^{(n+1)}_{\frac{2k+1}{2^{n+1}}}$, and $B^{(n+1)}_{\frac{2k+1}{2^{n+1}}} - B^{(n)}_{\frac{k}{2^n}} \sim \mathcal{N}(0, \frac{1}{2^{n+1}})$.

Proof. By induction. For $n = 0$. Let us first check that $B^{(1)}_{\frac{1}{2}} - B^{(0)}_0$ is independent of $B^{(1)}_1 - B^{(1)}_{\frac{1}{2}}$. Note that

$$B^{(1)}_{\frac{1}{2}} - B^{(0)}_0 = \frac{1}{2}Z_1 + \frac{1}{2}Z_{\frac{1}{2}} = \frac{1}{\sqrt{2}}\frac{Z_1 + Z_{\frac{1}{2}}}{\sqrt{2}},$$

$$B^{(1)}_1 - B^{(1)}_{\frac{1}{2}} = \frac{1}{2}Z_1 - \frac{1}{2}Z_{\frac{1}{2}} = \frac{1}{\sqrt{2}}\frac{Z_1 - Z_{\frac{1}{2}}}{\sqrt{2}}.$$

Since $Z_1, Z_{\frac{1}{2}}$ are i.i.d. $\mathcal{N}(0,1)$, the Main Lemma (c.f. beginning of the proof) tells us that $B^{(1)}_{\frac{1}{2}} - B^{(0)}_0$ is independent of $B^{(1)}_1 - B^{(1)}_{\frac{1}{2}}$ and that $B^{(1)}_{\frac{1}{2}} - B^{(0)}_0$ is $\mathcal{N}(0, \frac{1}{2})$.

Now, let $n \geq 1$, and assume that the property holds for $n-1$. We have

$$\begin{aligned}
B^{(n+1)}_{\frac{2k+1}{2^{n+1}}} - B^{(n)}_{\frac{k}{2^n}} &= \frac{1}{2}\left(B^{(n)}_{\frac{k}{2^n}} + B^{(n)}_{\frac{k+1}{2^n}}\right) + \frac{1}{2^{\frac{n+2}{2}}} Z_{\frac{2k+1}{2^{n+1}}} - B^{(n)}_{\frac{k}{2^n}} \\
&= \frac{1}{2}B^{(n)}_{\frac{k+1}{2^n}} - \frac{1}{2}B^{(n)}_{\frac{k}{2^n}} + \frac{1}{2^{\frac{n+2}{2}}} Z_{\frac{2k+1}{2^{n+1}}} \\
&= \frac{1}{2}\frac{1}{\sqrt{2^n}}\left[\sqrt{2^n}\left(B^{(n)}_{\frac{k+1}{2^n}} - B^{(n)}_{\frac{k}{2^n}}\right) + Z_{\frac{2k+1}{2^{n+1}}}\right].
\end{aligned}$$

By induction, $\left(B^{(n)}_{\frac{k+1}{2^n}} - B^{(n)}_{\frac{k}{2^n}}\right) \sim \mathcal{N}(0, \frac{1}{2^n})$. Also, $\left(B^{(n)}_{\frac{k+1}{2^n}} - B^{(n)}_{\frac{k}{2^n}}\right)$ and $Z_{\frac{2k+1}{2^{n+1}}}$ are independent (since the Z_q's are independent). Thus, by the Main Lemma again,

$$\frac{1}{\sqrt{2}}\left[\sqrt{2^n}\left(B^{(n)}_{\frac{k+1}{2^n}} - B^{(n)}_{\frac{k}{2^n}}\right) + Z_{\frac{2k+1}{2^{n+1}}}\right]$$

is standard Gaussian. It follows that $B^{(n+1)}_{\frac{2k+1}{2^{n+1}}} - B^{(n)}_{\frac{k}{2^n}} \sim \mathcal{N}(0, \frac{1}{2^{n+1}})$. Similarly, noting that

$$B^{(n)}_{\frac{k+1}{2^n}} - B^{(n+1)}_{\frac{2k+1}{2^{n+1}}} = \frac{1}{\sqrt{2^{n+1}}}\frac{1}{\sqrt{2}}\left[\sqrt{2^n}\left(B^{(n)}_{\frac{k+1}{2^n}} - B^{(n)}_{\frac{k}{2^n}}\right) - Z_{\frac{2k+1}{2^{n+1}}}\right],$$

we deduce the result by the Main Lemma again. □

Lemma 1.5. For all $n \geq 0$, for all $p < q \in \mathcal{D}_n$,

1. $B_q^{(n)} - B_p^{(n)} \sim \mathcal{N}(0, q-p)$.
2. $B_q^{(n)} - B_p^{(n)}$ is independent of $B_r^{(n)}$, for all $r \leq p$, $r \in \mathcal{D}_n$.

Proof. This is a consequence of Lemma 1.4.

1. Let $p, q \in \mathcal{D}_n$. Then there exists $k < l$ such that $p = \frac{k}{2^n}$ and $q = \frac{l}{2^n}$. Hence,

$$B_q^{(n)} - B_p^{(n)} = B_{\frac{l}{2^n}}^{(n)} - B_{\frac{k}{2^n}}^{(n)} = B_{\frac{l}{2^n}}^{(n)} - B_{\frac{l-1}{2^n}}^{(n)} + \cdots + B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{k}{2^n}}^{(n)}.$$

One can see that each term of sum are mutually independent (proof similar to Lemma 1.4). By Lemma 1.4 each term is a Gaussian $\mathcal{N}(0, \frac{1}{2^n})$, hence $B_q^{(n)} - B_p^{(n)} \sim \mathcal{N}(0, q-p)$.

2. Same argument. □

Lemma 1.6. Lemma 1.5 holds for all $p < q \in \mathcal{D}$.

Proof. If $p, q \in \mathcal{D}$, then there exists $n \in \mathbb{N}$ such that $p, q \in \mathcal{D}_n$. Apply then Lemma 1.5. □

Step 2: [Almost sure uniform convergence]

Let us denote, for $\omega \in \Omega$,

$$\Delta^{(n)}(\omega) = \max_{t\in[0,1]} |B_t^{(n+1)}(\omega) - B_t^{(n)}(\omega)| = \max_{k\in\{0,\ldots,2^n-1\}} \max_{t\in[\frac{k}{2^n},\frac{k+1}{2^n}]} |B_t^{(n+1)}(\omega) - B_t^{(n)}(\omega)|.$$

We drop the variable ω next. Since, by definition, $B_t^{(n)}$ is defined by linear interpolation and $B_t^{(n+1)} = B_t^{(n)}$ when $t \in \mathcal{D}_n$, we see that

$$\max_{t\in[\frac{k}{2^n},\frac{k+1}{2^n}]} |B_t^{(n+1)} - B_t^{(n)}|$$

is attained at the midpoint $t = \frac{2k+1}{2^{n+1}}$ (draw a picture). Hence,

$$\begin{aligned}\Delta^{(n)} &= \max_{k\in\{0,\ldots,2^n-1\}} |B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{2k+1}{2^{n+1}}}^{(n)}| = \max_{k\in\{0,\ldots,2^n-1\}} |B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - \frac{1}{2}\left(B_{\frac{k}{2^n}}^{(n)} + B_{\frac{k+1}{2^n}}^{(n)}\right)| \\ &= \frac{1}{2}\max_{k\in\{0,\ldots,2^n-1\}} |\left(B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{k}{2^n}}^{(n)}\right) - \left(B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{2k+1}{2^{n+1}}}^{(n+1)}\right)|.\end{aligned}$$

Note that $B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{k}{2^n}}^{(n)}$ and $B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{2k+1}{2^{n+1}}}^{(n+1)}$ are i.i.d. Gaussian $\mathcal{N}(0, \frac{1}{2^{n+1}})$. Hence, for all k,

$$W_k^{(n)} = \left(B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{k}{2^n}}^{(n)}\right) - \left(B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{2k+1}{2^{n+1}}}^{(n+1)}\right)$$

is Gaussian $\mathcal{N}(0, \frac{1}{2^n})$. Let $\alpha \geq 1$. One has,

$$\mathbb{P}(\Delta^{(n)} \geq \frac{\alpha}{2\sqrt{2^n}}) = \mathbb{P}(\frac{1}{2}\max_{k\in\{0,\ldots,2^n-1\}} |W_k^{(n)}| \geq \frac{\alpha}{2\sqrt{2^n}}) \leq 2^n \mathbb{P}(\frac{1}{2}|W_0^{(n)}| \geq \frac{\alpha}{2\sqrt{2^n}}),$$

where the inequality comes from the union bound. Note that for $Z \sim \mathcal{N}(0,1)$,

$$\mathbb{P}(Z \geq \alpha) \leq \frac{e^{-\frac{\alpha^2}{2}}}{\alpha\sqrt{2\pi}},$$

hence, by symmetry of Gaussian and the fact that $\sqrt{2^n}W_0^{(n)} \sim \mathcal{N}(0,1)$,

$$\mathbb{P}(\Delta^{(n)} \geq \frac{\alpha}{2\sqrt{2^n}}) = 2^{n+1}\mathbb{P}(\sqrt{2^n}W_0^{(n)} \geq \alpha) \leq 2^{n+1}\frac{e^{-\frac{\alpha^2}{2}}}{\alpha\sqrt{2\pi}}.$$

Now, take $\alpha = 2\sqrt{n}$. Then,

$$\mathbb{P}(\Delta^{(n)} \geq \frac{\sqrt{n}}{\sqrt{2^n}}) \leq \frac{1}{\sqrt{2\pi n}}\left(\frac{2}{e^2}\right)^n.$$

Hence,

$$\sum_{n\geq 1}\mathbb{P}(\Delta^{(n)} \geq \frac{\sqrt{n}}{\sqrt{2^n}}) < +\infty.$$

By Borel-Cantelli,

$$\mathbb{P}(\limsup\{\Delta^{(n)} \geq \frac{\sqrt{n}}{\sqrt{2^n}}\}) = 0.$$

In other words, there exists $A \subset \Omega$, $\mathbb{P}(A) = 1$, such that for all $\omega \in \Omega$, there exists $N \in \mathbb{N}$, such that for all $n \geq N$,

$$\Delta^{(n)}(\omega) \leq \frac{\sqrt{n}}{\sqrt{2^n}}.$$

Recalling the definition of $\Delta^{(n)}(\omega)$, we thus proved that for all ω in a set A of probability 1,

$$\sum_{n\geq 1}\|B^{n+1}(\omega) - B^n(\omega)\|_{L^\infty([0,1])} < +\infty.$$

A standard result of analysis allows us to conclude that, almost surely, $\{B^{(n)}(\omega)\}_{n\geq 1}$ converges uniformly on $[0,1]$. We then define

$$B(\omega) = \begin{cases} \lim_{n\to+\infty} B^{(n)}(\omega) & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}.$$

Step 3: [The limit is a Brownian motion on $[0,1]$]

- **Continuity:** By construction, for all $\omega \in \Omega$, for all $n \in \mathbb{N}$, $t \mapsto B_t^{(n)}(\omega)$ is continuous on $[0,1]$. Since, almost surely, $\{B_n\}$ converges uniformly on $[0,1]$ to B, we deduce that, almost surely, $t \mapsto B_t(\omega)$ is continuous.
- Since for all $n \in \mathbb{N}$, $B_0^{(n)} = 0$, we deduce that $B_0 = 0$.
- **Stationary increments:** Let $t, s \in \mathcal{D}$. Then, there exists $m \in \mathbb{N}$ such that $t, s \in \mathcal{D}_m$. Hence, $B_t^{(m)} - B_s^{(m)} \sim \mathcal{N}(0, t-s)$. By construction, for all $t \in \mathcal{D}_m$, for all $n \geq m$, $B_t^{(n)} = B_t^{(m)}$. Hence
$$B_t - B_s = \lim_{n\to+\infty} B_t^{(n)} - B_s^{(n)} = \lim_{n\to+\infty} B_t^{(m)} - B_s^{(m)} = B_t^{(m)} - B_s^{(m)},$$
where the limit is understood as "almost sure convergence". Since $B_t^{(m)} - B_s^{(m)} \sim \mathcal{N}(0, t-s)$, we have $B_t - B_s \sim \mathcal{N}(0, t-s)$. Now, assume that $t, s \in [0,1]$. By density of $\mathcal{D}$ in $[0,1]$, there exist sequences $\{t_k\}, \{s_k\} \in \mathcal{D}$ such that $t = \lim_k t_k$ and $s = \lim_k s_k$. Since, almost surely, $t \mapsto B_t$ is continuous, we have, almost surely, $B_t = \lim_k B_{t_k}$ and $B_s = \lim_k B_{s_k}$. Since, for all k, $B_{t_k} - B_{s_k} \sim \mathcal{N}(0, t_k - s_k)$, we can conclude that $B_t - B_s \sim \mathcal{N}(0, t-s)$ (use, for example, characteristic functions).
- **Independent increments:** Same argument.

1.3 Simulation of Brownian motion

Fix an integer $n \in \mathbb{N}$. Given times $0 = t_0 < t_1 < \cdots < t_n$, generate $Z_1, \ldots, Z_n$ i.i.d. $\mathcal{N}(0,1)$. Define

$$\begin{aligned}
B_0 &= 0, \\
B_{t_1} &= \sqrt{t_1} Z_1, \\
B_{t_2} &= B_{t_1} + \sqrt{t_2 - t_1} Z_2 = \sqrt{t_1} Z_1 + \sqrt{t_2 - t_1} Z_2, \\
&\vdots \\
B_{t_n} &= B_{t_{n-1}} + \sqrt{t_n - t_{n-1}} Z_n = \sum_{i=1}^{n} \sqrt{t_i - t_{i-1}} Z_i
\end{aligned}$$

Using this construction, $\{B_t\}$ is a Brownian motion at times $0 = t_0 < t_1 < \cdots < t_n$. Indeed, it starts at 0, and for all $l \leq m < n$,

$$B_{t_n} - B_{t_m} = \sum_{i=1}^{n} \sqrt{t_i - t_{i-1}} Z_i - \sum_{i=1}^{m} \sqrt{t_i - t_{i-1}} Z_i = \sum_{i=m+1}^{n} \sqrt{t_i - t_{i-1}} Z_i,$$

which is Gaussian $\mathcal{N}(0, t_n - t_m)$, and is independent of B_{t_l}.

1.4 Properties of the Brownian motion

Definition 1.7. $\{X_t\}_{t \geq 0}$ is a Gaussian process if for all $n \in \mathbb{N}$, for all $t_1 < \cdots < t_n$, the random vector $(X_{t_1}, \ldots, X_{t_n})$ is multivariate Gaussian.

Theorem 1.1. $(X_1, \ldots, X_n)$ is multivariate Gaussian $\iff$ every linear combination of the X_i's is Gaussian, that is, for all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $\lambda_1 X_1 + \cdots + \lambda_n X_n$ is Gaussian $\iff$

$$\exists\, \mu \in \mathbb{R}^n, \exists\, A \in \mathbb{R}^{n \times m}, (X_1, \ldots, X_n) = \mu + A(Z_1, \ldots, Z_n),$$

where $Z_1, \ldots, Z_n$ are i.i.d. $\mathcal{N}(0,1)$.

Theorem 1.2. A Brownian motion is a Gaussian process.

Proof. Define

$$Z_j = \frac{B_{t_j} - B_{t_{j-1}}}{\sqrt{t_j - t_{j-1}}}, \quad j = 1, \ldots, n.$$

In particular, the Z_j's are i.i.d. standard Gaussian $\mathcal{N}(0,1)$. Note that

$$\begin{pmatrix} B_{t_1} \\ \vdots \\ \vdots \\ B_{t_n} \end{pmatrix} = \begin{pmatrix} \sqrt{t_1} & 0 & \cdots & 0 \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & \cdots & \sqrt{t_n - t_{n-1}} \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ \vdots \\ Z_n \end{pmatrix}.$$

Hence $\{B_t\}$ is a Gaussian process. □

Definition 1.8. Let $\{\mathcal{F}_t\}$ be a filtration. The germ σ-algebra is

$$\mathcal{F}_s^+ = \cap_{t > s} \mathcal{F}_t.$$

Remark 1.9.

1. In general $\mathcal{F}_s^+ \neq \mathcal{F}_s$. Indeed, let X be a non-constant random variable. Define $X_t = tX$, $t \geq 0$, and $\mathcal{F}_t = \sigma(X_s : 0 \leq s \leq t)$. Note that for all $t > 0$, $\mathcal{F}_t = \sigma(X)$. However,

$$\{\emptyset, \Omega\} = \mathcal{F}_0 \neq \cap_{t > 0} \mathcal{F}_t = \sigma(X).$$

2. $\mathcal{F}_s^+$ represents an infinitesimal additional information into the future.

Theorem 1.3 (Blumenthal 0-1 Law)**.** Let $\{B_t\}$ be a Brownian motion. If $A \in \mathcal{F}_0^+$, then $\mathbb{P}(A) = 0$ or 1.

Corollary 1.10. Let $\{B_t\}$ be a standard Brownian motion. Define

$$T_1 = \inf\{t > 0 : B_t > 0\}, \qquad T_2 = \inf\{t > 0 : B_t = 0\}, \qquad T_3 = \inf\{t > 0 : B_t < 0\}.$$

Then, $\mathbb{P}(T_1 = 0) = \mathbb{P}(T_2 = 0) = \mathbb{P}(T_3 = 0) = 1$.

Proof. One has

$$\{T_1 = 0\} = \cap_{n\geq 1} \cup_{\varepsilon\in(0,\frac{1}{n})} \{B_\varepsilon > 0\}.$$

Hence, $\{T_1 = 0\} \in \mathcal{F}_0^+$. Note that for all $t > 0$,

$$\{B_t > 0\} \subset \{T_1 \leq t\},$$

hence

$$\mathbb{P}(T_1 \leq t) \geq \mathbb{P}(B_t > 0) = \frac{1}{2}.$$

We deduce that

$$\mathbb{P}(T_1 = 0) = \mathbb{P}(\cap_n\{T_1 \leq \frac{1}{n}\}) = \lim_n \mathbb{P}(T_1 \leq \frac{1}{n}) \geq \frac{1}{2}.$$

Since $\{T_1 = 0\} \in \mathcal{F}_0^+$, by Blumenthal 0-1 law, we conclude that $\mathbb{P}(T_1 = 0) = 1$.

By symmetry, (that is, $\{-B_t\}$ is a Brownian motion), $\mathbb{P}(T_3 = 0) = 1$.

With probability 1, $t \mapsto B_t$ is continuous and satisfies $\mathbb{P}(T_1 = 0) = \mathbb{P}(T_3 = 0) = 1$, hence by the intermediate value theorem, $\mathbb{P}(T_2 = 0) = 1$. □

Remark 1.11. In particular, Corollary 1.10 tells us that with proba 1, for all $\varepsilon > 0$, B_t hits 0 infinitely many times in the interval $(0, \varepsilon)$.

Theorem 1.4 (Long term behavior of Brownian motion)**.** Let $\{B_t\}$ be a Brownian motion, then

$$\limsup_{t\to+\infty} \frac{B_t}{\sqrt{t}} = +\infty \text{ and } \liminf_{t\to+\infty} \frac{B_t}{\sqrt{t}} = -\infty.$$

Proof. Fix $M > 0$.

$$\mathbb{P}(\limsup_{t\to+\infty} \frac{B_t}{\sqrt{t}} > M) = \mathbb{P}(\limsup_{t\to 0^+} \sqrt{t}B_{\frac{1}{t}} > M) = \mathbb{P}(\cap_{t>0} \cup_{0\leq s\leq t} \{\sqrt{s}B_{\frac{1}{s}} > M\})$$

Fact: $\{sB_{\frac{1}{s}}\}$ is a Brownian motion (Time inversion — see later).
Fact: $\{\limsup f_t > M\} = \limsup\{f_t > M\}$.

Note that $\sqrt{s}B_{\frac{1}{s}} = \frac{X_s}{\sqrt{s}}$, where $X_s = sB_{\frac{1}{s}}$ being a Brownian motion. Hence,

$$\cap_{t>0} \cup_{0\leq s\leq t} \{\sqrt{s}B_{\frac{1}{s}} > M\} = \cap_{t>0} \cup_{0\leq s\leq t} \{X_s > M\sqrt{s}\} \in \mathcal{F}_0^+.$$

By Blumenthal 0-1 law,

$$\mathbb{P}(\limsup_{t\to+\infty} \frac{B_t}{\sqrt{t}} > M) = 0 \text{ or } 1.$$

Now, note that

$$\mathbb{P}(\limsup_{t\to+\infty} \frac{B_t}{\sqrt{t}} > M) \geq \mathbb{P}(\limsup_{n\to+\infty} \frac{B_n}{\sqrt{n}} > M) = \mathbb{P}(\cap_{n\geq 1} \cup_{k\geq n} \{\frac{B_k}{\sqrt{k}} > M\})$$

$$= \lim_{n\to+\infty} \mathbb{P}(\cup_{k\geq n}\{\frac{B_k}{\sqrt{k}} > M\}) \geq \lim_{n\to+\infty} \mathbb{P}(\{\frac{B_n}{\sqrt{n}} > M\}) = \lim_{n\to+\infty} \mathbb{P}(\{B_1 > M\}) = \mathbb{P}(B_1 > M) > 0.$$

We conclude that

$$\mathbb{P}(\limsup_{t\to+\infty}\frac{B_t}{\sqrt{t}} > M) = 1.$$

It follows that

$$\mathbb{P}(\limsup_{t\to+\infty}\frac{B_t}{\sqrt{t}} = +\infty) = \mathbb{P}(\cap_{M>0}\limsup_{t\to+\infty}\frac{B_t}{\sqrt{t}} > M) = \lim_{M\to+\infty}\mathbb{P}(\limsup_{t\to+\infty}\frac{B_t}{\sqrt{t}} > M) = 1.$$

By symmetry ($\{-B_t\}$ is a Brownian motion), we deduce that

$$\mathbb{P}(\liminf_{t\to+\infty}\frac{B_t}{\sqrt{t}} = -\infty) = 1.$$

□

Remark 1.12. In other words, a Brownian motion is recurrent (each value $a \in \mathbb{R}$ is visited infinitely many often).

Definition 1.13. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Recall that a filtration $\{F_t\}_{t\geq 0}$ is a familly of sigma-algebras such that for all $s \leq t$, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$.

A process $\{M_t\}_{t\geq 0}$ is a $\{F_t\}$ continuous-time martingale if

i) For all $t \geq 0$, M_t is $\mathcal{F}_t$-measurable.
ii) For all $t \geq 0$, $\mathbb{E}[|M_t|] < +\infty$.
iii) For all $s \leq t$, $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$.

Proposition 1.14. A Brownian motion is a continuous-time martingale.

Proof.

$$\mathbb{E}[B_t|\mathcal{F}_s] = \mathbb{E}[B_t - B_s + B_s|\mathcal{F}_s] = \mathbb{E}[B_t - B_s|\mathcal{F}_s] + B_s = B_s,$$

because $B_t - B_s$ is independent of $\mathcal{F}_s$ and has expectation 0. □

Theorem 1.5 (Law of Large Numbers for Brownian motion)**.** For a Brownian motion $\{B_t\}$, $\lim_{t\to+\infty}\frac{B_t}{t} = 0$ almost surely.

Proof. **Step 1:** Note that $B_n = B_1 - B_0 + \cdots + B_n - B_{n-1}$, so we can write

$$B_n = \sum_{k=1}^{n} X_k,$$

where $X_k = B_k - B_{k-1}$. Note that $\{X_k\}$ is a sequence of i.i.d. $\mathcal{N}(0,1)$ random variable. Hence, by the strong LLN, $\frac{B_n}{n} \to \mathbb{E}[B_1] = 0$ almost surely.

Step 2: We will prove that

$$\sum_{n\geq 0}\mathbb{P}(\sup_{t\in[n,n+1]}|B_t - B_n| > n^{\frac{2}{3}}) < +\infty.$$

Fix $n \geq 0$. Define, for $m \geq 0$ and $k \in \{0, \ldots, 2^m\}$,

$$X_k = B_{n+\frac{k}{2^m}} - B_n.$$

Since $\{B_{n+t} - B_n\}_{t\geq 0}$ is a Brownian motion, it is a martingale. It follows that $\{X_k\}$ is a discrete time martingale. We can thus apply Doob's inequality and obtain

$$\mathbb{P}(\sup_{0\leq k\leq 2^m}|X_k| > n^{\frac{2}{3}}) \leq \frac{\mathbb{E}[|X_{2^m}|^2]}{n^{\frac{4}{3}}} = \frac{\mathbb{E}[|B_{n+1} - B_n|^2]}{n^{\frac{4}{3}}} = \frac{1}{n^{\frac{4}{3}}}.$$

Because $t \mapsto B_t$ is continuous, we have

$$\{\sup_{t\in[n,n+1]} |B_t - B_n| > n^{\frac{2}{3}}\} = \cup_{m\geq 1}\{\sup_{0\leq k\leq 2^m} |X_k| > n^{\frac{2}{3}}\}.$$

Hence,

$$\mathbb{P}(\sup_{t\in[n,n+1]} |B_t - B_n| > n^{\frac{2}{3}}) = \lim_{m\to+\infty} \mathbb{P}(\sup_{0\leq k\leq 2^m} |X_k| > n^{\frac{2}{3}}) \leq \frac{1}{n^{\frac{4}{3}}}.$$

Step 3: Define, for $n \geq 0$,

$$A_n = \{\sup_{t\in[n,n+1]} |B_t - B_n| > n^{\frac{2}{3}}\}.$$

Since $\sum \mathbb{P}(A_n) < +\infty$, by Borel-Cantelli we have $\mathbb{P}(\limsup A_n) = 0$. This means that, for all ω in a set of probability 1,

$$\exists n_0 \geq 1, \forall n \geq n_0, \forall t \in [n, n+1], \left|\frac{B_t(\omega)}{t}\right| \leq \frac{n}{t}\left(\left|\frac{B_t(\omega) - B_n(\omega)}{n}\right| + \left|\frac{B_n(\omega)}{n}\right|\right) \leq \frac{1}{n^{\frac{1}{3}}} + \left|\frac{B_n(\omega)}{n}\right|,$$

which goes to 0 as $n \to +\infty$. □

Corollary 1.15 (Time Inversion)**.** Let $\{B_t\}$ be a Brownian motion. The process $\{X_t\}_{t\geq 0}$ defined by $X_t = tB_{\frac{1}{t}}$ for $t > 0$ and $X_0 = 0$, is a Brownian motion, for the natural filtration $\widetilde{\mathcal{F}}_t = \sigma(X_s : s \leq t)$.

Proof. **Continuity at 0:** From Theorem 1.5, we have

$$\lim_{t\to 0^+} X_t = \lim_{t\to+\infty} X_{\frac{1}{t}} = \lim_{t\to+\infty} \frac{B_t}{t} = 0 = X_0.$$

Gaussian Increments: Note that, for $s \leq t$,

$$X_t - X_s = (t-s)B_{\frac{1}{t}} - s(B_{\frac{1}{s}} - B_{\frac{1}{t}}),$$

which is $\mathcal{N}(0, t-s)$.

Independent Increments: Since $(X_t - X_s, X_s)$ is a bivariate Gaussian, we can conclude independence because $\mathbb{E}[(X_t - X_s)X_s] = 0$. □

1.5 Reflection Principle

Definition 1.16. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ be a filtered space. Let $\{X_t\}$ be a stochastic process adapted to $\{\mathcal{F}_t\}$. We say that $\{X_t\}$ is a Markov process if

$$\forall A \in \mathcal{F}, \forall h \geq 0, \forall t \geq 0, \quad \mathbb{P}(X_{t+h} \in A|\mathcal{F}_t) = \mathbb{P}(X_{t+h} \in A|X_t).$$

Notation:

$$\mathbb{P}(X_{t+h} \in A|X_t) = \mathbb{P}(X_{t+h} \in A|\sigma(X_t)) = \mathbb{E}[1_A(X_{t+h})|\sigma(X_t)].$$

Theorem 1.6. A Brownian motion is a Markov process (w.r.t the same filtration).

Sketch of proof. We want to prove that

$$\mathbb{P}(B_{t+h} \in A|\mathcal{F}_t) = \mathbb{P}(B_{t+h} \in A|B_t),$$

equivalently,

$$\mathbb{E}[1_A(B_{t+h})|\mathcal{F}_t] = \mathbb{E}[1_A(B_{t+h})|\sigma(B_t)].$$

Let $\Phi\colon \mathbb{R} \to \mathbb{R}$ be measurable, then

$$\mathbb{E}[\Phi(B_{t+h})|\mathcal{F}_t] = \mathbb{E}[\Phi(B_{t+h} - B_t + B_t)|\mathcal{F}_t] = \mathbb{E}[g(X, B_t)|\mathcal{F}_t],$$

where $X = B_{t+h} - B_t$, which is independent of $\mathcal{F}_t$, and $g(x, y) = \Phi(x + y)$.

Since X is independent of $\mathcal{F}_t$, and B_t is $\sigma(B_t)$-measurable, $\mathbb{E}[g(X, B_t)|\mathcal{F}_t] = \mathbb{E}[g(X, B_t)|\sigma(B_t)]$. To prove this, start with functions g of the form $g(x, y) = 1_C(x)1_D(y)$, and use the fact that they approximate any Borel function. □

Definition 1.17. A random variable T is an $\{\mathcal{F}_t\}$-stopping time if

$$\forall t \geq 0, \quad \{T \leq t\} \in \mathcal{F}_t.$$

Proposition 1.18. 1. Every deterministic time is a stopping time.

2. If $\{T_n\}$ is a sequence of stopping time, the $\sup_n T_n$ is a stopping time.

Proof. 1. Exercise.
2. Fix $t \geq 0$. Then,

$$\{\sup_n T_n \leq t\} = \cap_n \{T_n \leq t\} \in \mathcal{F}_t.$$

□

Remark 1.19. In general, $\inf_n T_n$ is not a stopping time. Indeed, recalling that if $m = \inf(A)$, then for all $\varepsilon > 0$, there exists $a \in A$, such that $m \geq a - \varepsilon$. In particular we have

$$\{\inf_n T_n \leq t\} = \cap_{\varepsilon>0} \cup_{n\geq 1} \{T_n \leq t + \varepsilon\} \in \cap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t^+.$$

Since in general $\mathcal{F}_t \neq \mathcal{F}_t^+$, it follows that $\inf_n T_n$ is not a stopping time.

Similarly, note that, when $\mathcal{F}_t = \sigma(B_s : s \leq t)$,
1. If F is a closed set, then $T = \inf\{t \geq 0 : B_t \in F\}$ is a stopping time.
2. If O is open, then $T = \inf\{t \geq 0 : B_t \in O\}$ is not a stopping time.

Definition 1.20. A filtration $\{F_t\}$ is right-continuous if for all $t \geq 0$, $\mathcal{F}_t = \mathcal{F}_t^+$.

Example 1.21. The canonical filtration for a Brownian motion $\{B_t\}$:
Define

$$\mathcal{F}_t = \sigma(B_s : s \leq t), \quad t \geq 0,$$

and

$$\widetilde{\mathcal{F}}_t = \mathcal{F}_t^+ = \cap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}, \quad t \geq 0.$$

Then $\{\widetilde{\mathcal{F}}_t\}$ is a right-continuous filtration, and $\{B_t\}$ is adapted to $\{\widetilde{\mathcal{F}}_t\}$.

Proposition 1.22. 1. If $\{T_n\}$ is a sequence of $\{\mathcal{F}_t^+\}$-stopping times, then $\inf_n T_n$ is an $\{\mathcal{F}_t^+\}$-stopping time.

2. If O is open, then $T = \inf\{t \geq 0 : B_t \in O\}$ is an $\{\widetilde{\mathcal{F}}_t\}$-stopping time.

Definition 1.23. For a stopping time T, define

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t^+, \forall\, t \geq 0\}.$$

Theorem 1.7. $\mathcal{F}_T$ is a σ-algebra.

Proof. Same proof as in the discrete case. □

Definition 1.24. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ be a filtered space. Let $\{X_t\}$ be a stochastic process adapted to $\{\mathcal{F}_t\}$. We say that $\{X_t\}$ is a strong Markov process if for all stopping time T finite almost surely,

$$\forall A \in \mathcal{F}, \forall h \geq 0, \quad \mathbb{P}(X_{T+h} \in A | \mathcal{F}_T) = \mathbb{P}(X_{T+h} \in A | X_T).$$

Theorem 1.8. The Brownian motion is a strong Markov process.

Sketch of Proof. Note that $\{B_{T+t} - B_T\}_{t\geq 0}$ is a standard Brownian motion independent of $\mathcal{F}_T$.
□

Theorem 1.9 (Reflection principle)**.** Let T be a stopping time and $\{B_t\}$ be a standard Brownian motion.

If $M = (x, y)$, then the reflection of M with respect to the line passing through $(0, a)$ and parallel to the x-axis is $M^* = (x, 2a - y)$ (draw a picture).

For $t \geq 0$, define

$$B_t^* = B_t 1_{t \leq T} + (2B_T - B_t) 1_{t > T}.$$

Then, $\{B_t^*\}$ is a standard Brownian motion.

Definition 1.25. The process B_t^* defined in Theorem 1.9 is called reflected Brownian motion.

Corollary 1.26. Let $\{B_t\}$ be a Brownian motion. Consider, for $t \geq 0$,

$$M_t = \sup_{0 \leq s \leq t} B_s.$$

Then, $M_t \sim |Z|$, where $Z \sim \mathcal{N}(0, t)$. This means that supremum of Brownian motion path has a χ distribution.

Proof. First note that $\mathbb{P}(M_t \geq 0) = 1$ because $B_0 = 0$ a.s.

Fix $a > 0$. Let us find $\mathbb{P}(M_t \geq a)$. Consider $\{B_t^*\}$ the reflected Brownian motion with respect to the stopping time $T_a = \inf\{t \geq 0 : B_t = a\}$. Note that
i) $\{B_t \geq a\} \subset \{M_t \geq a\}$.
ii) $\{M_t \geq a\} \cap \{B_t < a\} = \{B_t^* > a\}$.
The point i) is clear. The inclusion $\subset$ of the point ii) is clear from the picture (after reflection, $B_t < a$ if and only if $B_t^* > a$). For the other inclusion $\supset$, if $\{B_t^* > a\}$, then either $\{B_t > a\}$ either $\{B_t < a\}$. The case $\{B_t > a\}$ is impossible because $B_t > a$ implies that $T_a < t$. Necessarily, $\{B_t < a\}$. Since $\{B_t < a\}$ and $\{B_t^* > a\}$, we have $T_a \leq t$ and thus $M_t \geq a$.

Thus, from ii) and Theorem 1.9,

$$\mathbb{P}(M_t \geq a, B_t < a) = \mathbb{P}(B_t^* > a) = \mathbb{P}(B_t > a).$$

Hence,

$$\mathbb{P}(M_t \geq a) = \mathbb{P}(M_t \geq a, B_t < a) + \mathbb{P}(M_t \geq a, B_t \geq a) = 2\mathbb{P}(B_t \geq a) = \mathbb{P}(|B_t| \geq a).$$

□

1.6 Differentiability of the paths of the Brownian motion

Theorem 1.10. With probability 1, the paths of the Brownian motion are nowhere differentiable. Formally, let $\{B_t\}$ be a Brownian motion, then

$$\mathbb{P}\Big(\{\omega \in \Omega : \exists t_0 \in [0, +\infty), t \mapsto B_t(\omega) \text{ is differentiable at } t_0\}\Big) = 0.$$

Proof. **Step 1: [Setup]**

Without loss of generality, let us prove the result on $[0, 1]$. Denote

$$A = \{\omega \in \Omega : \exists t_0 \in [0, 1], t \mapsto B_t(\omega) \text{ is differentiable at } t_0\}.$$

We want to prove that $\mathbb{P}(A) = 0$. For $n \geq 3$ and $k \in \{1, \ldots, n-2\}$, define

$$M_{k,n} = \max\{|B_{\frac{k+2}{n}} - B_{\frac{k+1}{n}}|, |B_{\frac{k+1}{n}} - B_{\frac{k}{n}}|, |B_{\frac{k}{n}} - B_{\frac{k-1}{n}}|\},$$

and

$$M_n = \min(M_{1,n}, \ldots, M_{n-2,n}).$$

Step 2: The goal of Step 2 is to prove that

$$\forall \omega \in A, \exists M \in \mathbb{N}, \exists n_0 \in N, \forall n \geq n_0, M_n(\omega) \leq \frac{5}{n}(1+M).$$

Let $\omega \in A$ (we drop the dependence on ω next). Then there exists $t_0 \in [0,1]$ such that $t \mapsto B_t$ is differentiable at t_0. By definition of differentiability, there exists $L \in \mathbb{R}$ and $\delta > 0$ such that for all $t \in [0,1] \setminus \{t_0\}$, if $|t - t_0| \leq \delta$, then $|B_t - B_{t_0} - L(t-t_0)| \leq |t - t_0|$ (taking $\varepsilon = 1$). Hence, by triangular inequality, for all t such that $|t - t_0| \leq \delta$,

$$|B_t - B_{t_0}| \leq (1 + |L|)|t - t_0|.$$

Now, note that there exists $n_0 \geq 1$ and $k \in \{1, \dots, n_0\}$, such that

$$t_0 \in \left[\frac{k-1}{n_0}, \frac{k}{n_0}\right] \text{ and } \left|\frac{k+2}{n_0} - \frac{k-1}{n_0}\right| = \frac{3}{n_0} \leq \delta.$$

Let $n \geq n_0$. Then there exists $k \in \{1, \dots, n\}$ such that the above holds. Hence,

$$|B_{\frac{k}{n}} - B_{\frac{k-1}{n}}| \leq |B_{\frac{k}{n}} - B_{t_0}| + |B_{t_0} - B_{\frac{k-1}{n}}| \leq (1+|L|)\left(|\frac{k}{n} - t_0| + |t_0 - \frac{k-1}{n}|\right) \leq \frac{2}{n}(1+|L|).$$

Similarly, we have

$$|B_{\frac{k+1}{n}} - B_{\frac{k}{n}}| \leq \frac{3}{n}(1+|L|) \quad \text{and} \quad |B_{\frac{k+2}{n}} - B_{\frac{k+1}{n}}| \leq \frac{5}{n}(1+|L|).$$

We thus proved that for all $n \geq n_0$, there exists $k \in \{1, \dots, n\}$ such that

$$M_{k,n} \leq \frac{5}{n}(1+|L|).$$

By definition of M_n, this tells us that for all $n \geq n_0$,

$$M_n \leq \frac{5}{n}(1+|L|).$$

Now, just take any integer M greater than $|L|$ to conclude that

$$\forall \omega \in A, \exists M \in \mathbb{N}, \exists n_0 \in N, \forall n \geq n_0, M_n(\omega) \leq \frac{5}{n}(1+M).$$

Equivalently,

$$A \subset \cup_{M \in \mathbb{N}} \cup_{n_0 \in \mathbb{N}} \cap_{n \geq n_0} \left\{M_n \leq \frac{5}{n}(1+M)\right\}.$$

Step 3: The goal of Step 3 is to prove that

$$\forall M \in \mathbb{N}, \lim_{n \to +\infty} \mathbb{P}(M_n \leq \frac{5}{n}(1+M)) = 0.$$

Let $n \geq 3$ and $k \in \{1, \dots, n-2\}$. Denote,

$$X_1 = |B_{\frac{k}{n}} - B_{\frac{k-1}{n}}|, \quad X_2 = |B_{\frac{k+1}{n}} - B_{\frac{k}{n}}|, \quad X_3 = |B_{\frac{k+2}{n}} - B_{\frac{k+1}{n}}|.$$

Since $\{B_t\}$ is a Brownian motion, X_1, X_2, X_3 are i.i.d. with same distribution as $|Z|$ where $Z \sim \mathcal{N}(0, \frac{1}{n})$. Thus, the CDF of $M_{k,n} = \max(X_1, X_2, X_3)$ is

$$F_{M_{k,n}}(x) = \mathbb{P}(M_{k,n} \leq x) = \mathbb{P}(X_1 \leq x)^3, \quad x \in \mathbb{R}.$$

Note that

$$\mathbb{P}(X_1 \leq x) = \mathbb{P}(|Z| \leq x\sqrt{n}),$$

where $Z \sim \mathcal{N}(0,1)$. Hence,

$$\mathbb{P}(X_1 \leq x) = \frac{2}{\sqrt{2\pi}} \int_0^{x\sqrt{n}} e^{-\frac{t^2}{2}}\, dt \leq \frac{2x\sqrt{n}}{\sqrt{2\pi}}.$$

We deduce that for all $M \in \mathbb{N}$,

$$\mathbb{P}(M_{k,n} \leq \frac{5}{n}(1+M)) \leq \left[\frac{10}{\sqrt{2\pi}}(1+M)\frac{1}{\sqrt{n}}\right]^3 = \frac{C}{n^{\frac{3}{2}}},$$

where $C = \left[\frac{10}{\sqrt{2\pi}}(1+M)\right]^3$. Hence, by union bound,

$$\mathbb{P}\Bigg(M_n \leq \frac{5}{n}(1+M)\Bigg) = \mathbb{P}\Bigg(\cup_{k=1}^{n-2}\left\{M_{k,n} \leq \frac{5}{n}(1+M)\right\}\Bigg) \leq \sum_{k=1}^{n-2}\mathbb{P}\Bigg(M_{k,n} \leq \frac{5}{n}(1+M)\Bigg) \leq \frac{C}{\sqrt{n}}.$$

We conclude that

$$\forall M \in \mathbb{N},\ \lim_{n\to+\infty} \mathbb{P}(M_n \leq \frac{5}{n}(1+M)) = 0.$$

Step 4: [Conclusion]
From Step 2,

$$\mathbb{P}(A) \leq \mathbb{P}\Bigg(\cup_{M\in\mathbb{N}} \cup_{n_0\in\mathbb{N}} \cap_{n\geq n_0} \left\{M_n \leq \frac{5}{n}(1+M)\right\}\Bigg).$$

Denote $B_{n_0} = \cap_{n\geq n_0}\{M_n \leq \frac{5}{n}(1+M)\}$, and note that $\{B_{n_0}\}$ is an increasing sequence of sets, hence, from Step 3,

$$\begin{aligned}
\mathbb{P}\Bigg(\cup_{n_0\in\mathbb{N}} \cap_{n\geq n_0}\left\{M_n \leq \frac{5}{n}(1+M)\right\}\Bigg) &= \lim_{n_0\to+\infty} \mathbb{P}\Bigg(\cap_{n\geq n_0}\left\{M_n \leq \frac{5}{n}(1+M)\right\}\Bigg) \\
&\leq \lim_{n_0\to+\infty} \mathbb{P}\Bigg(\left\{M_{n_0} \leq \frac{5}{n_0}(1+M)\right\}\Bigg) \\
&= 0.
\end{aligned}$$

We conclude that

$$\mathbb{P}(A) \leq \sum_{M\in\mathbb{N}} \mathbb{P}\Bigg(\cup_{n_0\in\mathbb{N}} \cap_{n\geq n_0}\left\{M_n \leq \frac{5}{n}(1+M)\right\}\Bigg) = 0.$$

□