1 Convergence of Martingales

Recall that $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ is a filtered space.

1.1 Uniform Integrability

Remark 1.1.

$$\begin{aligned} X \text{ integrable} &\iff \mathbb{E}[|X|] < +\infty \\ &\iff \forall \varepsilon > 0,\, \exists \delta > 0,\, \mathbb{E}[|X|1_{\{|X|>\delta\}}] \leq \varepsilon. \end{aligned}$$

Definition 1.2. (Uniform integrability) Let $\{X_i\}_{i\in I}$ be a family of random variables. We say that $\{X_i\}$ is uniformly integrable (or equi-integrable) if

$$\forall \varepsilon > 0,\, \exists \delta > 0,\, \forall i \in I,\, \mathbb{E}[|X_i|1_{\{|X_i|>\delta\}}] \leq \varepsilon.$$

Example 1.3. Let $M \in L^1(\mathcal{F})$. Then, $M_n = \mathbb{E}[M|\mathcal{F}_n]$, $n \geq 1$, is a uniformly integrable martingale (Exercise).

Proposition 1.4. If $\{X_i\}$ is uniformly integrable, then $\{X_i\}$ is tight.

Proof. Recall that $\{X_i\}$ is tight if

$$\forall \varepsilon > 0,\, \exists K \subset \mathbb{R} \text{ compact},\, \forall i \in I,\, \mathbb{P}(X_i \in K) > 1 - \varepsilon.$$

Let $\varepsilon > 0$. Since $\{X_i\}$ is uniformly integrable, then

$$\exists \delta > 0,\, \forall i \in I,\, \mathbb{E}[|X|1_{\{|X|>\delta\}}] \leq \varepsilon.$$

Choose $\delta' = \max(\delta, 1)$. Thus, for all $i \in I$,

$$\delta'\mathbb{P}(|X_i| > \delta') \leq \mathbb{E}[|X_i|1_{\{|X_i|>\delta'\}}] \leq \mathbb{E}[|X_i|1_{\{|X_i|>\delta\}}] \leq \varepsilon.$$

Choose $K = [-\delta', \delta']$, then

$$\mathbb{P}(X_i \notin K) \leq \frac{\varepsilon}{\delta'} \leq \varepsilon.$$

$\square$

Remark 1.5. The converse is false (Exercise).

1.2 Convergence Theorem

Theorem 1.1. For $\{M_n\}$ martingale, the following are equivalent:

1. $\{M_n\}$ is uniformly integrable.
2. $\{M_n\}$ converges in L^1 and a.s.
3. $\{M_n\}$ converges in L^1.

Remark 1.6. For martingales, convergence in L^1 is stronger than convergence a.s.

Before proving Theorem 1.1, we establish intermediate lemmas.

Lemma 1.7. If $\{X_n\}$ is uniformly integrable, then $\{X_n\}$ is bounded in L^1 (that is, $\sup_n \mathbb{E}[|X_n|] < +\infty$).

Proof. Fix $\varepsilon > 0$, then there exists $\delta > 0$ such that for all n,

$$\mathbb{E}[|X_n|] = \mathbb{E}[|X_n|1_{\{|X_n|\leq\delta\}}] + \mathbb{E}[|X_n|1_{\{|X_n|>\delta\}}] \leq \delta + \varepsilon.$$

$\square$

Lemma 1.8. If $\{X_n\}$ is uniformly bounded (that is, $\exists K$, $\forall n$, $|X_n| \leq K$ a.s.) and if $X_n \to X$ in probability, then $X_n \to X$ in L^1.

Proof. • X is bounded by K. Indeed,

$$\forall m \geq 1,\ \{X \geq K + \frac{1}{m}\} \subset \{|X_n - X| \geq \frac{1}{m}\} \quad \text{(Exercise)}.$$

Hence,

$$\forall m,\ \mathbb{P}(X \geq K + \frac{1}{m}) \leq \mathbb{P}(|X_n - X| \geq \frac{1}{m}) \to_{n\to+\infty} 0.$$

Thus,

$$\forall m,\ \mathbb{P}(X \geq K + \frac{1}{m}) = 0,$$

which implies that $|X| \leq K$ a.s.

• Now fix $\varepsilon > 0$. We have

$$\mathbb{E}[|X_n - X|] = \mathbb{E}[|X_n - X|1_{\{|X_n - X|>\frac{\varepsilon}{2}\}}] + \mathbb{E}[|X_n - X|1_{\{|X_n - X|\leq\frac{\varepsilon}{2}\}}].$$

Since $|X_n - X| \leq |X_n| + |X| \leq 2K$ a.s., we have

$$\mathbb{E}[|X_n - X|1_{\{|X_n - X|>\frac{\varepsilon}{2}\}}] + \mathbb{E}[|X_n - X|1_{\{|X_n - X|\leq\frac{\varepsilon}{2}\}}] \leq 2K\,\mathbb{P}(|X_n - X| > \frac{\varepsilon}{2}) + \frac{\varepsilon}{2} \leq \varepsilon,$$

for n large enough. This shows L^1 convergence. $\square$

Lemma 1.9. Let $\{X_n\}$ be uniformly integrable. If $X_n \to X$ a.s., then $X_n \to X$ in L^1.

Proof. • X is in L^1. Indeed, by Fatou's lemma,

$$\mathbb{E}[|X|] = \mathbb{E}[\liminf |X_n|] \leq \liminf \mathbb{E}[|X_n|] \leq \sup_n \mathbb{E}[|X_n|],$$

which is finite by Lemma 1.7.

• Fix $K > 0$. Let $\Phi_K(x) = \text{sgn}(x)\min(|x|, K)$, where $\text{sgn}(x) = 1$ if $x \geq 0$ and $\text{sgn}(x) = -1$ if $x < 0$. Note that if $|X| \leq K$ then $\Phi_K(X) = \text{sgn}(X)|X| = X$. We deduce that

$$\Phi_K(X)1_{\{|X|\leq K\}} = X1_{\{|X|\leq K\}}.$$

If $|X| > K$, then $\Phi_K(X) = \text{sgn}(X)K$, and note that in this case

$$|\text{sgn}(X)K - X| \leq |X|.$$

We deduce that

$$|\Phi_K(X) - X| \leq |X| 1_{\{|X|>K\}}.$$

Hence,

$$\begin{aligned}
\mathbb{E}[|X_n - X|] &\leq \mathbb{E}[|\Phi_K(X_n) - X_n|] + \mathbb{E}[|\Phi_K(X) - X|] + \mathbb{E}[|\Phi_K(X_n) - \Phi_K(X)|] \\
&\leq \mathbb{E}[|X_n| 1_{\{|X_n|>K\}}] + \mathbb{E}[|X| 1_{\{|X|>K\}}] + \mathbb{E}[|\Phi_K(X_n) - \Phi_K(X)|].
\end{aligned}$$

Since $\{X_n\}$ is uniformly integrable, the first term can be made $\leq \frac{\varepsilon}{3}$ by choosing K sufficiently large. Since $X \in L^1$, the second term can also be made $\leq \frac{\varepsilon}{3}$ for K sufficiently large. The last term can be made $\leq \frac{\varepsilon}{3}$ by using Lemma 1.8 ($\Phi_K(X_n)$ is uniformly bounded and converges to $\Phi_K(X)$ a.s. and thus in probability). Finally,

$$\mathbb{E}[|X_n - X|] \leq \varepsilon,$$

which proves that $X_n \to X$ in L^1. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. 1. $\implies$ 2.: Assume $\{M_n\}$ uniformly integrable. From Lemma 1.7, $\{M_n\}$ is thus bounded in L^1. From Doob's lemma (theorem previously discussed in class), it follows that $\{M_n\}$ converges a.s. It thus follows, from Lemma 1.9, that $\{M_n\}$ converges in L^1.

2. $\implies$ 3.: Immediate.

3. $\implies$ 1.: Assume that $\{M_n\}$ converges to M in L^1. Since $M_n \in L^1$, we have $M \in L^1$, and thus there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|\mathbb{E}[|M_n|] - \mathbb{E}[|M|]| \leq \mathbb{E}[|M_n - M|] \leq 1,$$

and thus, for all $n \geq N$,

$$\mathbb{E}[|M_n|] \leq 1 + \mathbb{E}[|M|].$$

We deduce that $\{M_n\}$ is bounded in L^1, that is $\sup_n \mathbb{E}[|M_n|] < +\infty$.

Now, fix $\varepsilon > 0$. Choose $\delta' > 0$ such that for all $A \in \mathcal{F}$, $\mathbb{P}(A) \leq \delta' \implies \mathbb{E}[|M| 1_A] \leq \frac{\varepsilon}{2}$ (possible by Lebesgue theorem). Take $\delta = \frac{\sup_n \mathbb{E}[|M_n|]}{\delta'}$. By Markov's inequality, for all $n \geq 1$, $\mathbb{P}(|M_n| > \delta) \leq \delta'$, and thus, for all n large enough,

$$\mathbb{E}[|M_n| 1_{\{|M_n|>\delta\}}] \leq \mathbb{E}[|M_n - M| 1_{\{|M_n|>\delta\}}] + \mathbb{E}[|M| 1_{\{|M_n|>\delta\}}] \leq \varepsilon.$$

We conclude that $\{M_n\}$ is uniformly integrable. □