Monday, March 23

1 Poisson Process

Recall that $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ is a filtered space.

Definition 1.1. (Poisson Process)
A stochastic process $\{N_t\}_{t\in[0,+\infty)}$ is a Poisson process with parameter λ (with respect to $\{\mathcal{F}_t\}$) if

1. $N_0 = 0$.
2. With probability 1, the paths of $\{N_t\}$ are right continuous with left limits.
3. For all $0 \le s < t$, $N_t - N_s$ follows a Poisson distribution of parameter $\lambda(t-s)$ (stationary increments). In short, we write $N_t - N_s \sim \mathcal{P}(\lambda(t-s))$.
4. For all $0 \le s < t$, $N_t - N_s$ is independent of $\mathcal{F}_s$ (independent increments).

Remark 1.2. 1. Recall that the paths (or trajectories) of a stochastic process $\{N_t\}$ is the function $t \mapsto N_t(\omega)$, for fixed $\omega \in \Omega$.

2. The property 2. means that $\exists A \subset \Omega$, $\mathbb{P}(A) = 0$, $\forall \omega \in \Omega \setminus A$, the function $t \mapsto N_t(\omega)$ is right continuous function with left limits on $[0, +\infty)$.
3. From 3., taking $s = 0$, we have that for all $t \ge 0$, $N_t \sim \mathcal{P}(\lambda t)$.

Theorem 1.1. Let $\{N_t\}$ be a Poisson process. Then, with proba 1, the paths of $\{N_t\}$ are increasing and are constant, except for jumps of size 1.

Proof. • For all $s, t \ge 0$, if $s < t$, then $N_t - N_s \sim \mathcal{P}(\lambda(t-s))$. Hence,

$$\mathbb{P}(N_t - N_s \in \mathbb{N}) = 1,$$

which implies that

$$\mathbb{P}(N_t \ge N_s) = 1.$$

Caution: Here, we did **not** prove that, with proba 1, the paths of $\{N_t\}$ are increasing. We proved that, for all $0 \le s < t$,

$$\mathbb{P}(N_t \ge N_s) = 1. \tag{1}$$

Instead, we want

$$\mathbb{P}(\forall\, 0 \le s < t,\ N_t \ge N_s) = 1,$$

which is much stronger. (can you see why?)
Formally, we want to prove that

$$\exists A \subset \Omega,\ \mathbb{P}(A) = 0,\ \forall \omega \in \Omega \setminus A,\ \forall\, 0 \le s < t,\ N_s(\omega) \le N_t(\omega).$$

The set of probability 1 (in particular ω) does **not** depend on s, t, while in (1), it does.

So far, we proved

$$\forall\, 0 \leq s < t,\ \exists A \subset \Omega,\ \mathbb{P}(A) = 0,\ \forall \omega \in \Omega \setminus A,\ N_s(\omega) \leq N_t(\omega).$$

Here, A (and ω) depends on s, t.

For $s < t$, define

$$A_{s,t} = \{\omega \in \Omega : N_s(\omega) \leq N_t(\omega)\}.$$

We proved that for all $s < t$, $\mathbb{P}(A_{s,t}) = 1$. Define

$$A = \cap_{s,t \in \mathbb{Q}_+,\, s<t} A_{s,t}.$$

Since the union is countable, and $\mathbb{P}(A_{s,t}) = 1$, it follows that $\mathbb{P}(A) = 1$ (Exercise). Hence, we proved that

$$\mathbb{P}(\,\forall s, t \in \mathbb{Q}_+,\ 0 \leq s < t,\ N_t \geq N_s) = 1,$$

that is

$$\exists A \subset \Omega,\ \mathbb{P}(A) = 0,\ \forall \omega \in \Omega \setminus A,\ \forall s, t \in \mathbb{Q}_+\ 0 \leq s < t \Longrightarrow N_s(\omega) \leq N_t(\omega). \tag{2}$$

Since, with proba 1, $t \mapsto N_t$ is right continuous, we deduce that (2) holds for all $s, t \in [0, +\infty)$.

- For all $t \geq 0$, $N_t \sim \mathcal{P}(\lambda t)$, and thus

$$\mathbb{P}(N_t \in \mathbb{N}) = 1.$$

Using same argument as above, we derive that

$$\mathbb{P}(\forall\, t \geq 0,\ N_t \in \mathbb{N}) = 1.$$

Wednesday, March 25

- For $t > 0$, let us denote $N_{t^-} = \lim_{s \to t_{s<t}}$, the left limit at t. If N_t is continuous at t, then $N_t - N_{t-} = 0$. Fix $T > 0$. Define

$$A_T = \{w \in \Omega : \exists\, t < T,\ N_t(\omega) - N_{t-}(\omega) \geq 2\}.$$

Fix $n \geq 1$, and define

$$B_n = \{w \in \Omega : \exists\, k \in \{0, \ldots, n-1\}, N_{\frac{(k+1)T}{n}}(\omega) - N_{\frac{kT}{n}}(\omega) \geq 2\}.$$

Note that for all $n \geq 1$, $A_T \subset B_n$ (draw a picture). This implies that for all $n \geq 1$, $\mathbb{P}(A_T) \leq \mathbb{P}(B_n)$, and thus

$$\mathbb{P}(A_T) \leq \lim_{n \to +\infty} \mathbb{P}(B_n).$$

Note that

$$\mathbb{P}(B_n) = \mathbb{P}(\,\exists\, k \in \{0, \ldots, n-1\}, N_{\frac{(k+1)T}{n}} - N_{\frac{kT}{n}} \geq 2) \leq \sum_{k=0}^{n-1} \mathbb{P}(N_{\frac{(k+1)T}{n}} - N_{\frac{kT}{n}} \geq 2),$$

by the union bound. Since

$$N_{\frac{(k+1)T}{n}} - N_{\frac{kT}{n}} \sim \mathcal{P}\left(\lambda\left(\frac{(k+1)T}{n} - \frac{kT}{n}\right)\right) = \mathcal{P}\left(\lambda\frac{T}{n}\right),$$

we deduce that

$$\mathbb{P}(N_{\frac{(k+1)T}{n}} - N_{\frac{kT}{n}} \geq 2) = \mathbb{P}(N_{\frac{T}{n}} \geq 2),$$

which is independent of k. Recall that for a Poisson distribution N of parameter μ,

$$\mathbb{P}(N \geq 2) = 1 - \mathbb{P}(N = 0) - \mathbb{P}(N = 1) = 1 - \mathrm{e}^{-\mu} - \mu\mathrm{e}^{-\mu}.$$

Taking $\mu = \lambda\frac{T}{n}$, we have

$$\mathbb{P}(B_n) \leq n\left(1 - \mathrm{e}^{-\lambda\frac{T}{n}} - \lambda\frac{T}{n}\mathrm{e}^{-\lambda\frac{T}{n}}\right) = n\left(1 - \mathrm{e}^{-\lambda\frac{T}{n}}\right) - \lambda T\mathrm{e}^{-\lambda\frac{T}{n}}.$$

We conclude that $\lim_n \mathbb{P}(B_n) = 0$, and thus

$$\mathbb{P}(A_T) = 0.$$

Since $T > 0$ was arbitrary, we proved that for all $T > 0$, $\mathbb{P}(A_T) = 0$. Define

$$A = \{\exists\, t \geq 0,\ N_t - N_{t-} \geq 2\},$$

and recall that we want to prove that $\mathbb{P}(A) = 0$. Note that

$$A = \cup_{T>0, T\in\mathbb{Q}} A_T.$$

Hence, from above, and the fact that the union is countable, $\mathbb{P}(A) = 0$. □

Interpretation: The Poisson process is used for modeling the time of arrivals in a system (study of queues). N_t represents, for example, the number of customers who entered a store at time t, or the number of phone calls received at time t.

Simulation: To simulate a Poisson process, we use the following theorem (the proof is left as an exercise).

Theorem 1.2. Let $\{X_n\}$ be a sequence of i.i.d. random variables. Define, for $n \geq 1$,

$$S_n = X_1 + \cdots + X_n,$$

and define, for $t \geq 0$,

$$N_t = \sum_{n\geq 1} 1_{\{S_n \leq t\}}.$$

Then, the following are equivalent:

1. X_1 is distributed accordingly to an exponential distribution of parameter $\lambda > 0$.
2. $\{N_t\}$ is a Poisson process of parameter $\lambda > 0$.

- We deduce that to simulate a Poisson process, we can perform the following:

1. Fix $n \geq 1$. Generate $U_1, \ldots, U_n$ random numbers in $[0, 1]$.
2. Fix $\lambda > 0$. From $i = 1$ to n, put
$$X_i = \frac{-\log(1 - U_i)}{\lambda},$$
so that X_i's are exponentially distributed.
3. For $i = 1$ to n, put $S_i = X_1 + \cdots + X_i$.
4. Fix $T > 0$. For $t \in [0, T]$, put $N_t = k$ if $S_k \leq t < S_{k+1}$.

Friday, March 27

The Bus Paradox:

Question: At bus stop #2, a bus is scheduled to arrive every 15 minutes. You need to take that bus, but forgot to check the schedule. How long are you going to wait for the bus, in average?

Of course, in reality, you cannot predict, with 100% accuracy, when a bus is going to come. The answer strongly depends on how you model the arrivals.

To determine the distribution of the arrivals, you could perform statistical tests, but this requires a lot of time. Instead, let us consider the following model (which is usually considered enough for this type of problem):

Model: We assume that the arrivals of the bus is a Poisson process $\{N_t\}$ of parameter $\lambda = \frac{1}{15}$ (a bus comes every 15 minutes). As surprising as it is, the answer will be 15 minutes!

N_t represents the number of arrivals at time t. Using Theorem 1.2, it will be useful to consider the inter-arrivals. Let $\{X_n\}$ be a sequence of i.i.d. random variables with an exponential distribution of parameter λ. Denote, for $n \geq 1$,

$$S_n = X_1 + \cdots + X_n.$$

From Theorem 1.2, for $t \geq 0$,

$$N_t = \sum_{n \geq 1} 1_{\{S_n \leq t\}}.$$

Note that

$$\{N_t = k\} = \{S_k \leq t < S_{k+1}\},$$
$$\{N_t \geq k\} = \{S_k \leq t\}.$$

Define, for $t \geq 0$,

$$Z_t = t - S_{N_t},$$

with the convention that $Z_t = t$ if $N_t = 0$. Define also

$$W_t = S_{N_t+1} - t.$$

Note that W_t represents the waiting time before the next bus. Hence, we need to find $\mathbb{E}[W_t]$. For that, we are going to find the distribution of (Z_t, W_t). Fix $z \in [0, t]$ and $w > 0$. If $\{Z_t \geq z\}$ and $\{W_t > w\}$, then there are no arrivals between the times $[t - z, t + w]$. Hence,

$$\{Z_t \geq z\} \cap \{W_t > w\} = \{N_{t-z} = N_{t+w}\} = \{N_{t+w} - N_{t-z} = 0\}.$$

We deduce that

$$\mathbb{P}(Z_t \geq z, W_t > w) = \mathbb{P}(N_{t+w} - N_{t-z} = 0) = \mathbb{P}(N_{w+z} = 0) = \mathrm{e}^{-\lambda(w+z)} = \mathrm{e}^{-\lambda w}\mathrm{e}^{-\lambda z}.$$

It follows that

$$\mathbb{P}(W_t > w) = \mathbb{P}(Z_t \geq 0, W_t > w) = \mathrm{e}^{-\lambda w}.$$

We conclude that W_t follows an exponential distribution of parameter λ. Hence, the average waiting time is

$$\mathbb{E}[W_t] = \frac{1}{\lambda} = 15 \text{ minutes}.$$