1 Brownian motion

Recall that a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ is given.

1.1 Definitions

Definition 1.1 (Standard Brownian motion). A continuous-time stochastic process $\{B_t\}_{t \in [0, +\infty)}$ is a standard Brownian motion if

1. $B_0 = 0$ a.s.
2. $\forall 0 \leq s < t, B_t - B_s \sim \mathcal{N}(0, t - s)$ (Gaussian of mean 0 and variance $t - s$).
3. $\forall 0 \leq s < t, B_t - B_s$ is independent of $\mathcal{F}_s$.
4. With probability 1, the trajectories are continuous. Precisely:

$$\exists A \subset \Omega, \mathbb{P}(A) = 1, \forall \omega \in \Omega, t \mapsto B_t(\omega)$$

is continuous on $[0, +\infty)$.

Remark 1.2. One may ask whether all the assumptions are necessary in the definition of the Brownian motion. Or, in other words, does one or several assumptions imply another one.

- The continuity assumption is a necessity. To see this, let $\{B_t\}$ be a Brownian motion and let $U$ be uniformly distributed on $[0, 1]$. Define, for $\omega \in \Omega$ and $t \geq 0$,

$$\tilde{B}_t(\omega) = B_t(\omega)1_{\{t \neq U(\omega)\}} + (1 + B_t(\omega))1_{\{t = U(\omega)\}}.$$

In this case, for all $t \geq 0$, $\mathbb{P}(\tilde{B}_t = B_t) = 1$, and hence $\tilde{B}_t$ satisfies properties 1-3. of the definition. However, for all $\omega \in \Omega$, $t \mapsto \tilde{B}_t(\omega)$ is discontinuous (at $t = U(\omega)$).

- It can be shown that if 3-4. hold, then 2. necessary hold.
- Property 1. is just a normalization. A brownian motion can start at any point.
- We will always consider the natural filtration $\mathcal{F}_t = \sigma(B_s : s \leq t)$.

Model: Brownian motions are used to model the trajectories of small particles in a fluid, or the evolution of the stock market. Generally speaking, it is used to model erratic motions.

Remark 1.3. When we say “Let $\{B_t\}_{t \geq 0}$ be a Brownian motion”, we implicitly assume the existence of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a family of random variables $\{B_t\}$ on $(\Omega, \mathcal{F})$ such that $\mathbb{P}$ makes $\{B_t\}$ a Brownian motion (that is, such that $\{B_t\}$ satisfies the definitions 1-4. with respect to $\mathbb{P}$).

Question: Does such a probability space exist?

Answer: Yes, but technical to prove. This is the goal of the next section.
1.2 Construction of the Brownian motion

We will restrict the construction to \([0, 1]\). For \(n \geq 0\), denote
\[
\mathcal{D}_n = \left\{ \frac{k}{2^n} : k \in \{0, \ldots, 2^n\} \right\}.
\]
For example,
\[
D_0 = \{0, 1\}, \quad D_1 = \left\{ 0, \frac{1}{2}, 1 \right\}, \quad D_2 = \left\{ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \right\}.
\]
Denote
\[
\mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}_n,
\]
the dyadic of \([0, 1]\). Before starting, first note that \(\mathcal{D}\) is dense in \([0, 1]\), and that \(\{\mathcal{D}_n\}\) is increasing (\(\mathcal{D}_n \subset \mathcal{D}_{n+1}\)).

The process will follow the following steps:

**Step 1:** For each \(n \in \mathbb{N}\), build a continuous process \(\{B_t^{(n)}\}_{t \in [0, 1]}\) that satisfies the properties of the Brownian motion on \(\mathcal{D}_n\).

**Step 2:** With probability 1, \(t \mapsto B_t^{(n)}\) converges uniformly on \([0, 1]\).

**Step 3:** \(\lim_{n \to +\infty} B_t^{(n)}\) is a Brownian motion.

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**Step 1: [Construction on the dyadic]**

Let \(\{Z_q\}_{q \in \mathcal{D}}\) be a sequence of i.i.d. standard Gaussian. In particular, for all \(q \neq r \in \mathcal{D}\), \(Z_q\) is independent of \(Z_r\), and \(Z_q \sim \mathcal{N}(0, 1)\).

**Main Lemma:** If \(X, Y\) are i.i.d. \(\mathcal{N}(0, 1)\), then \(\frac{X+Y}{\sqrt{2}}\) and \(\frac{X-Y}{\sqrt{2}}\) are i.i.d. \(\mathcal{N}(0, 1)\).

**Proof:** Exercise.
Note that then, interpolate linearly to build Lemma 1.5. Since \( Z \)

Proof. For \( n \geq 0 \). Let us first check that \( B_1^{(1)} - B_0^{(0)} \) is independent of \( B_1^{(1)} - B_2^{(1)} \). We have

\[
B_1^{(1)} - B_0^{(0)} = \frac{1}{2} Z_1 + \frac{1}{2} Z_2 = \frac{1}{\sqrt{2}} \frac{Z_1 + Z_2}{\sqrt{2}}.
\]

Since \( Z_1, Z_2 \) are i.i.d. \( \mathcal{N}(0,1) \), the Main Lemma (c.f. beginning of the proof) tells us that \( B_1^{(1)} - B_0^{(0)} \) is independent of \( B_1^{(1)} - B_2^{(1)} \) and that \( B_1^{(1)} - B_0^{(0)} \) is \( \mathcal{N}(0, \frac{1}{2}) \).

For \( n \geq 1 \), and assume that the property holds for \( n-1 \). We have

\[
B_{2k+1}^{(n+1)} - B_k^{(n)} = \frac{1}{2} \left( B_k^{(n)} + B_{k+1}^{(n+1)} \right) + \frac{1}{2} \frac{Z_{2k+1}}{2^n} - B_k^{(n)} = \frac{1}{2} B_k^{(n)} - \frac{1}{2} B_{k+1}^{(n+1)} + \frac{1}{2} \frac{Z_{2k+1}}{2^n}.
\]

By induction, \( B_{k+1}^{(n+1)} - B_k^{(n)} \sim \mathcal{N}(0, \frac{1}{2^n}) \). Also, \( B_{k+1}^{(n+1)} - B_k^{(n)} \) and \( Z_{2k+1} \) are independent (since the \( Z_q \)'s are independent). Thus, by the Main Lemma again,

\[
\frac{1}{\sqrt{2}} \left[ \sqrt{2^n} \left( B_{k+1}^{(n)} - B_k^{(n)} \right) + Z_{2k+1} \right]
\]

is standard Gaussian. It follows that \( B_{2k+1}^{(n+1)} - B_{k}^{(n)} \sim \mathcal{N}(0, \frac{1}{2^n}) \). Similarly, noting that

\[
B_{2k+1}^{(n)} - B_{k}^{(n)} = \frac{1}{2} \left[ \sqrt{2^n} \left( B_{k}^{(n)} - B_{k+1}^{(n+1)} \right) - Z_{2k+1} \right],
\]

we deduce the result by the Main Lemma again.

\[\square\]

Lemma 1.5. For all \( n \geq 0 \), for all \( p < q \in \mathcal{D}_n \),

1. \( B_q^{(n)} - B_p^{(n)} \sim \mathcal{N}(0, q-p) \).
2. $B^{(n)}_q - B^{(n)}_p$ is independent of $B_r^{(n)}$, for all $r \leq p, r \in \mathcal{D}_n$.

**Proof.** This is a consequence of Lemma 1.4.

1. Let $p, q \in \mathcal{D}_n$. Then there exists $k < l$ such that $p = \frac{k}{2^n}$ and $q = \frac{l}{2^n}$. Hence,

$$B^{(n)}_q - B^{(n)}_p = B^{(n)}_{\frac{k}{2^n}} - B^{(n)}_{\frac{l}{2^n}} = B^{(n)}_{\frac{k}{2^n}} - B^{(n)}_{\frac{k+1}{2^n}} + \cdots + B^{(n)}_{\frac{l-1}{2^n}} - B^{(n)}_{\frac{l}{2^n}}.$$

One can see that each term of sum are mutually independent (proof similar to Lemma 1.4). By Lemma 1.4 each term is a Gaussian $\mathcal{N}(0, \frac{1}{2^n})$, hence $B^{(n)}_q - B^{(n)}_p \sim \mathcal{N}(0, q - p)$.

2. Same argument. □

**Lemma 1.6.** Lemma 1.5 holds for all $p < q \in \mathcal{D}$.

**Proof.** If $p, q \in \mathcal{D}$, then there exists $n \in \mathbb{N}$ such that $p, q \in \mathcal{D}_n$. Apply then Lemma 1.5. □

**Friday, April 3**

**Step 2: [Almost sure uniform convergence]**

Let us denote, for $\omega \in \Omega$,

$$\Delta^{(n)}(\omega) = \max_{t \in [0, 1]} |B_t^{(n+1)}(\omega) - B_t^{(n)}(\omega)| = \max_{k \in \{0, \ldots, 2^n-1\}} \max_{t \in [\frac{k}{2^n}, \frac{k+1}{2^n}]} |B_t^{(n+1)}(\omega) - B_t^{(n)}(\omega)|.$$

We drop the variable $\omega$ next. Since, by definition, $B_t^{(n)}$ is defined by linear interpolation and $B_t^{(n+1)} = B_t^{(n)}$ when $t \in \mathcal{D}_n$, we see that

$$\max_{t \in [\frac{k}{2^n}, \frac{k+1}{2^n}]} |B_t^{(n+1)} - B_t^{(n)}|$$

is attained at the midpoint $t = \frac{2k+1}{2^{n+1}}$ (draw a picture). Hence,

$$\Delta^{(n)} = \max_{k \in \{0, \ldots, 2^n-1\}} |B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{2k+1}{2^{n+1}}}^{(n)}| = \max_{k \in \{0, \ldots, 2^n-1\}} |B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - \frac{1}{2} \left( B_{\frac{k}{2^n}}^{(n)} + B_{\frac{k+1}{2^n}}^{(n)} \right) |$$

$$= \frac{1}{2} \max_{k \in \{0, \ldots, 2^n-1\}} \left| (B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{k}{2^n}}^{(n)}) - (B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{2k+1}{2^{n+1}}}^{(n+1)}) \right|,$$

Note that $B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{k}{2^n}}^{(n)}$ and $B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{2k+1}{2^{n+1}}}^{(n+1)}$ are i.i.d. Gaussian $\mathcal{N}(0, \frac{1}{2^{n+2}})$. Hence, for all $k$,

$$W_k^{(n)} = \left( B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{k}{2^n}}^{(n)} \right) - \left( B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} \right)$$

is Gaussian $\mathcal{N}(0, \frac{1}{2^n})$. Let $\alpha \geq 1$. One has,

$$\mathbb{P}(\Delta^{(n)} \geq \frac{\alpha}{2^{\sqrt{2^n}}}) = \mathbb{P}\left( \frac{1}{2} \max_{k \in \{0, \ldots, 2^n-1\}} |W_k^{(n)}| \geq \frac{\alpha}{2^{\sqrt{2^n}}} \right) \leq 2^n \mathbb{P}\left( \frac{1}{2} |W_0^{(n)}| \geq \frac{\alpha}{2^{\sqrt{2^n}}} \right),$$

4
where the inequality comes from the union bound. Note that for \( Z \sim \mathcal{N}(0, 1) \),

\[
P(Z \geq \alpha) \leq \frac{e^{-\frac{\alpha^2}{2\pi}}}{\alpha \sqrt{2\pi}},
\]

hence, by symmetry of Gaussian and the fact that \( \sqrt{2\pi}W_0^{(n)} \sim \mathcal{N}(0, 1) \),

\[
P(\Delta^{(n)} \geq \frac{\alpha}{2\sqrt{2n}}) = 2^{n+1}P(\sqrt{2\pi}W_0^{(n)} \geq \alpha) \leq 2^{n+1} \frac{e^{-\frac{\alpha^2}{2\sqrt{2\pi}}}}{\alpha \sqrt{2\pi}}.
\]

Now, take \( \alpha = 2\sqrt{n} \). Then,

\[
P(\Delta^{(n)} \geq \frac{\sqrt{n}}{\sqrt{2n}}) \leq \frac{1}{2\sqrt{2\pi n}} \left( \frac{2}{e^2} \right)^n.
\]

Hence,

\[
\sum_{n \geq 1} P(\Delta^{(n)} \geq \frac{\sqrt{n}}{\sqrt{2n}}) < +\infty.
\]

By Borel-Cantelli,

\[
P(\lim \sup \{ \Delta^{(n)} \geq \frac{\sqrt{n}}{\sqrt{2n}} \}) = 0.
\]

In other words, there exists \( A \subset \Omega, \ P(A) = 1 \), such that for all \( \omega \in \Omega \), there exists \( N \in \mathbb{N} \), such that for all \( n \geq N \),

\[
\Delta^{(n)}(\omega) \leq \frac{\sqrt{n}}{\sqrt{2n}}.
\]

Recalling the definition of \( \Delta^{(n)}(\omega) \), we thus proved that for all \( \omega \) in a set \( A \) of probability 1,

\[
\sum_{n \geq 1} \|B^{n+1}(\omega) - B^n(\omega)\|_{L^\infty([0,1])} < +\infty.
\]

A standard result of analysis allows us to conclude that, almost surely, \( \{B^{(n)}(\omega)\}_{n \geq 1} \) converges uniformly on \([0,1]\). We then define

\[
B(\omega) = \begin{cases} 
\lim_{n \to +\infty} B^{(n)}(\omega) & \text{if } \omega \in A \\
0 & \text{if } \omega \notin A.
\end{cases}
\]

Monday, April 6

Step 3: [The limit is a Brownian motion on \([0,1]\)]

Coming soon!