Monday, March 30

1 Brownian motion

Recall that a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ is given.

1.1 Definitions

Definition 1.1 (Standard Brownian motion)**.** A continuous-time stochastic process $\{B_t\}_{t\in[0,+\infty)}$ is a standard Brownian motion if

1. $B_0 = 0$ a.s.
2. $\forall\, 0 \le s < t$, $B_t - B_s \sim \mathcal{N}(0, t-s)$ (Gaussian of mean 0 and variance $t-s$).
3. $\forall\, 0 \le s < t$, $B_t - B_s$ is independent of $\mathcal{F}_s$.
4. With probability 1, the trajectories are continuous. Precisely:

$$\exists\, A \subset \Omega,\ \mathbb{P}(A) = 1, \forall\, \omega \in \Omega,\ t \mapsto B_t(\omega) \text{ is continuous on } [0, +\infty).$$

Remark 1.2. One may ask whether all the assumptions are necessary in the definition of the Brownian motion. Or, in other words, does one or several assumptions imply another one.

- The continuity assumption is a necessity. To see this, let $\{B_t\}$ be a Brownian motion and let U be uniformly distributed on $[0,1]$. Define, for $\omega \in \Omega$ and $t \ge 0$,

$$\widetilde{B}_t(\omega) = B_t(\omega)1_{\{t \neq U(\omega)\}} + (1 + B_t(\omega))1_{\{t = U(\omega)\}}.$$

In this case, for all $t \ge 0$, $\mathbb{P}(\widetilde{B}_t = B_t) = 1$, and hence $\widetilde{B}_t$ satisfies properties 1-3. of the definition. However, for all $\omega \in \Omega$, $t \mapsto \widetilde{B}_t(\omega)$ is discontinuous (at $t = U(\omega)$).

- It can be shown that if 3-4. hold, then 2. necessary hold.
- Property 1. is just a normalization. A brownian motion can start at any point.
- We will always consider the natural filtration $\mathcal{F}_t = \sigma(B_s : s \le t)$.

Model: Brownian motions are used to model the trajectories of small particles in a fluid, or the evolution of the stock market. Generally speaking, it is used to model erratic motions.

Remark 1.3. When we say "Let $\{B_t\}_{t\ge 0}$ be a Brownian motion", we implicitly assume the existence of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a family of random variables $\{B_t\}$ on $(\Omega, \mathcal{F})$ such that $\mathbb{P}$ makes $\{B_t\}$ a Brownian motion (that is, such that $\{B_t\}$ satisfies the definitions 1-4. with respect to $\mathbb{P}$).

Question: Does such a probability space exist?

Answer: Yes, but technical to prove. This is the goal of the next section.

Wednesday, April 1

1.2 Construction of the Brownian motion

We will restrict the construction to $[0,1]$. For $n \geq 0$, denote

$$\mathcal{D}_n = \left\{ \frac{k}{2^n} : k \in \{0, \dots, 2^n\} \right\}.$$

For example,

$$D_0 = \{0, 1\}, \quad D_1 = \left\{0, \frac{1}{2}, 1\right\}, \quad \mathcal{D}_2 = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}.$$

Denote

$$\mathcal{D} = \cup_{n \geq 0} D_n,$$

the dyadic of $[0,1]$. Before starting, first note that $\mathcal{D}$ is dense in $[0,1]$, and that $\{\mathcal{D}_n\}$ is increasing ($\mathcal{D}_n \subset \mathcal{D}_{n+1}$).

The process will follow the following steps:

Step 1: For each $n \in \mathbb{N}$, build a continuous process $\{B_t^{(n)}\}_{t \in [0,1]}$ that satisfies the properties of the Brownian motion on $\mathcal{D}_n$.

Step 2: With probability 1, $t \mapsto B_t^{(n)}$ converges uniformly on $[0,1]$.

Step 3: $\lim_{n \to +\infty} B_t^{(n)}$ is a Brownian motion.

Step 1: [Construction on the dyadic]

Let $\{Z_q\}_{q \in \mathcal{D}}$ be a sequence of i.i.d. standard Gaussian. In particular, for all $q \neq r \in \mathcal{D}$, Z_q is independent of Z_r, and $Z_q \sim \mathcal{N}(0,1)$.

Main Lemma: If X, Y are i.i.d. $\mathcal{N}(0,1)$, then $\frac{X+Y}{\sqrt{2}}$ and $\frac{X-Y}{\sqrt{2}}$ are i.i.d. $\mathcal{N}(0,1)$.

Proof: Exercise.

For each $\omega \in \Omega$, we are going to build $B_t^{(n)}(\omega)$ by induction on $n \in \mathbb{N}$, for $t \in \mathcal{D}_n$, and then interpolate <u>linearly</u>. We drop the variable ω next.

<u>**For $n = 0$:**</u>
Set $B_0^{(0)} = 0$ and $B_1^{(0)} = Z_1$. Then, we interpolate <u>linearly</u> between $B_0^{(0)}$ and $B_1^{(0)}$:

$$B_t^{(0)} = (1-t)B_0^{(0)} + tB_1^{(0)} = tZ_1, \quad t \in [0,1].$$

<u>**For $n = 1$:**</u>
Set

$$B_0^{(1)} = B_0^{(0)} = 0, \quad B_1^{(1)} = B_1^{(0)} = Z_1, \quad B_{\frac{1}{2}}^{(1)} = \frac{1}{2}\left(B_0^{(0)} + B_1^{(0)}\right) + \frac{1}{2} Z_{\frac{1}{2}} = \frac{1}{2} Z_1 + \frac{1}{2} Z_{\frac{1}{2}}.$$

Then, define $B_t^{(1)}$ by linear interpolation:

$$B_t^{(1)} = (1-2t)B_0^{(1)} + 2tB_{\frac{1}{2}}^{(1)} = 2tB_{\frac{1}{2}}^{(1)}, \quad t \in [0, \frac{1}{2}],$$

$$B_t^{(1)} = (2-2t)B_{\frac{1}{2}}^{(1)} + (2t-1)B_1^{(1)} \quad t \in [\frac{1}{2}, 1].$$

We continue this process for each $n \geq 0$.

For $n+1$:

Let $n \geq 0$. Assume $B_t^{(n)}$ built. For $k \in \{0, \ldots, 2^n - 1\}$, define

$$B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} = \frac{1}{2}\left(B_{\frac{k}{2^n}}^{(n)} + B_{\frac{k+1}{2^n}}^{(n)}\right) + \frac{1}{2^{\frac{n+2}{2}}} Z_{\frac{2k+1}{2^{n+1}}},$$

and for $t \in D_n$, define

$$B_t^{(n+1)} = B_t^{(n)}.$$

Then, interpolate linearly to build $B_t^{(n+1)}$ for all $t \in [0,1]$.

Lemma 1.4. For all $k \in \{0, \ldots, 2^n - 1\}$, $B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{k}{2^n}}^{(n)}$ is independent of $B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{2k+1}{2^{n+1}}}^{(n+1)}$, and $B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{k}{2^n}}^{(n)} \sim \mathcal{N}(0, \frac{1}{2^{n+1}})$.

Proof. By induction. For $n = 0$. Let us first check that $B_{\frac{1}{2}}^{(1)} - B_0^{(0)}$ is independent of $B_1^{(1)} - B_{\frac{1}{2}}^{(1)}$. Note that

$$B_{\frac{1}{2}}^{(1)} - B_0^{(0)} = \frac{1}{2}Z_1 + \frac{1}{2}Z_{\frac{1}{2}} = \frac{1}{\sqrt{2}}\frac{Z_1 + Z_{\frac{1}{2}}}{\sqrt{2}},$$

$$B_1^{(1)} - B_{\frac{1}{2}}^{(1)} = \frac{1}{2}Z_1 - \frac{1}{2}Z_{\frac{1}{2}} = \frac{1}{\sqrt{2}}\frac{Z_1 - Z_{\frac{1}{2}}}{\sqrt{2}}.$$

Since $Z_1, Z_{\frac{1}{2}}$ are i.i.d. $\mathcal{N}(0,1)$, the Main Lemma (c.f. beginning of the proof) tells us that $B_{\frac{1}{2}}^{(1)} - B_0^{(0)}$ is independent of $B_1^{(1)} - B_{\frac{1}{2}}^{(1)}$ and that $B_{\frac{1}{2}}^{(1)} - B_0^{(0)}$ is $\mathcal{N}(0, \frac{1}{2})$.

Now, let $n \geq 1$, and assume that the property holds for $n-1$. We have

$$\begin{aligned} B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{k}{2^n}}^{(n)} &= \frac{1}{2}\left(B_{\frac{k}{2^n}}^{(n)} + B_{\frac{k+1}{2^n}}^{(n)}\right) + \frac{1}{2^{\frac{n+2}{2}}} Z_{\frac{2k+1}{2^{n+1}}} - B_{\frac{k}{2^n}}^{(n)} \\ &= \frac{1}{2}B_{\frac{k+1}{2^n}}^{(n)} - \frac{1}{2}B_{\frac{k}{2^n}}^{(n)} + \frac{1}{2^{\frac{n+2}{2}}} Z_{\frac{2k+1}{2^{n+1}}} \\ &= \frac{1}{2}\frac{1}{\sqrt{2^n}}\left[\sqrt{2^n}\left(B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{k}{2^n}}^{(n)}\right) + Z_{\frac{2k+1}{2^{n+1}}}\right]. \end{aligned}$$

By induction, $\left(B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{k}{2^n}}^{(n)}\right) \sim \mathcal{N}(0, \frac{1}{2^n})$. Also, $\left(B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{k}{2^n}}^{(n)}\right)$ and $Z_{\frac{2k+1}{2^{n+1}}}$ are independent (since the Z_q's are independent). Thus, by the Main Lemma again,

$$\frac{1}{\sqrt{2}}\left[\sqrt{2^n}\left(B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{k}{2^n}}^{(n)}\right) + Z_{\frac{2k+1}{2^{n+1}}}\right]$$

is standard Gaussian. It follows that $B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{k}{2^n}}^{(n)} \sim \mathcal{N}(0, \frac{1}{2^{n+1}})$. Similarly, noting that

$$B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} = \frac{1}{\sqrt{2^{n+1}}}\frac{1}{\sqrt{2}}\left[\sqrt{2^n}\left(B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{k}{2^n}}^{(n)}\right) - Z_{\frac{2k+1}{2^{n+1}}}\right],$$

we deduce the result by the Main Lemma again. □

Lemma 1.5. For all $n \geq 0$, for all $p < q \in \mathcal{D}_n$,

1. $B_q^{(n)} - B_p^{(n)} \sim \mathcal{N}(0, q-p)$.

2. $B_q^{(n)} - B_p^{(n)}$ is independent of $B_r^{(n)}$, for all $r \le p$, $r \in \mathcal{D}_n$.

Proof. This is a consequence of Lemma 1.4.

1. Let $p, q \in \mathcal{D}_n$. Then there exists $k < l$ such that $p = \frac{k}{2^n}$ and $q = \frac{l}{2^n}$. Hence,

$$B_q^{(n)} - B_p^{(n)} = B_{\frac{l}{2^n}}^{(n)} - B_{\frac{k}{2^n}}^{(n)} = B_{\frac{l}{2^n}}^{(n)} - B_{\frac{l-1}{2^n}}^{(n)} + \cdots + B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{k}{2^n}}^{(n)}.$$

One can see that each term of sum are mutually independent (proof similar to Lemma 1.4). By Lemma 1.4 each term is a Gaussian $\mathcal{N}(0, \frac{1}{2^n})$, hence $B_q^{(n)} - B_p^{(n)} \sim \mathcal{N}(0, q-p)$.

2. Same argument. □

Lemma 1.6. Lemma 1.5 holds for all $p < q \in \mathcal{D}$.

Proof. If $p, q \in \mathcal{D}$, then there exists $n \in \mathbb{N}$ such that $p, q \in \mathcal{D}_n$. Apply then Lemma 1.5. □

Friday, April 3

Step 2: [Almost sure uniform convergence]

Let us denote, for $\omega \in \Omega$,

$$\Delta^{(n)}(\omega) = \max_{t\in[0,1]} |B_t^{(n+1)}(\omega) - B_t^{(n)}(\omega)| = \max_{k\in\{0,\dots,2^n-1\}} \max_{t\in[\frac{k}{2^n},\frac{k+1}{2^n}]} |B_t^{(n+1)}(\omega) - B_t^{(n)}(\omega)|.$$

We drop the variable ω next. Since, by definition, $B_t^{(n)}$ is defined by linear interpolation and $B_t^{(n+1)} = B_t^{(n)}$ when $t \in \mathcal{D}_n$, we see that

$$\max_{t\in[\frac{k}{2^n},\frac{k+1}{2^n}]} |B_t^{(n+1)} - B_t^{(n)}|$$

is attained at the midpoint $t = \frac{2k+1}{2^{n+1}}$ (draw a picture). Hence,

$$\Delta^{(n)} = \max_{k\in\{0,\dots,2^n-1\}} |B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{2k+1}{2^{n+1}}}^{(n)}| = \max_{k\in\{0,\dots,2^n-1\}} |B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - \frac{1}{2}\left(B_{\frac{k}{2^n}}^{(n)} + B_{\frac{k+1}{2^n}}^{(n)}\right)|$$

$$= \frac{1}{2} \max_{k\in\{0,\dots,2^n-1\}} |\left(B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{k}{2^n}}^{(n)}\right) - \left(B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{2k+1}{2^{n+1}}}^{(n+1)}\right)|.$$

Note that $B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{k}{2^n}}^{(n)}$ and $B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{2k+1}{2^{n+1}}}^{(n+1)}$ are i.i.d. Gaussian $\mathcal{N}(0, \frac{1}{2^{n+1}})$. Hence, for all k,

$$W_k^{(n)} = \left(B_{\frac{2k+1}{2^{n+1}}}^{(n+1)} - B_{\frac{k}{2^n}}^{(n)}\right) - \left(B_{\frac{k+1}{2^n}}^{(n)} - B_{\frac{2k+1}{2^{n+1}}}^{(n+1)}\right)$$

is Gaussian $\mathcal{N}(0, \frac{1}{2^n})$. Let $\alpha \ge 1$. One has,

$$\mathbb{P}(\Delta^{(n)} \ge \frac{\alpha}{2\sqrt{2^n}}) = \mathbb{P}(\frac{1}{2} \max_{k\in\{0,\dots,2^n-1\}} |W_k^{(n)}| \ge \frac{\alpha}{2\sqrt{2^n}}) \le 2^n \mathbb{P}(\frac{1}{2}|W_0^{(n)}| \ge \frac{\alpha}{2\sqrt{2^n}}),$$

where the inequality comes from the union bound. Note that for $Z \sim \mathcal{N}(0,1)$,

$$\mathbb{P}(Z \geq \alpha) \leq \frac{e^{-\frac{\alpha^2}{2}}}{\alpha\sqrt{2\pi}},$$

hence, by symmetry of Gaussian and the fact that $\sqrt{2^n}W_0^{(n)} \sim \mathcal{N}(0,1)$,

$$\mathbb{P}(\Delta^{(n)} \geq \frac{\alpha}{2\sqrt{2^n}}) = 2^{n+1}\mathbb{P}(\sqrt{2^n}W_0^{(n)} \geq \alpha) \leq 2^{n+1}\frac{e^{-\frac{\alpha^2}{2}}}{\alpha\sqrt{2\pi}}.$$

Now, take $\alpha = 2\sqrt{n}$. Then,

$$\mathbb{P}(\Delta^{(n)} \geq \frac{\sqrt{n}}{\sqrt{2^n}}) \leq \frac{1}{\sqrt{2\pi n}}\left(\frac{2}{e^2}\right)^n.$$

Hence,

$$\sum_{n\geq 1}\mathbb{P}(\Delta^{(n)} \geq \frac{\sqrt{n}}{\sqrt{2^n}}) < +\infty.$$

By Borel-Cantelli,

$$\mathbb{P}(\limsup\{\Delta^{(n)} \geq \frac{\sqrt{n}}{\sqrt{2^n}}\}) = 0.$$

In other words, there exists $A \subset \Omega$, $\mathbb{P}(A) = 1$, such that for all $\omega \in \Omega$, there exists $N \in \mathbb{N}$, such that for all $n \geq N$,

$$\Delta^{(n)}(\omega) \leq \frac{\sqrt{n}}{\sqrt{2^n}}.$$

Recalling the definition of $\Delta^{(n)}(\omega)$, we thus proved that for all ω in a set A of probability 1,

$$\sum_{n\geq 1}\|B^{n+1}(\omega) - B^n(\omega)\|_{L^\infty([0,1])} < +\infty.$$

A standard result of analysis allows us to conclude that, almost surely, $\{B^{(n)}(\omega)\}_{n\geq 1}$ converges uniformly on $[0,1]$. We then define

$$B(\omega) = \begin{cases} \lim_{n\to+\infty} B^{(n)}(\omega) & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}.$$

Monday, April 6

Step 3: [The limit is a Brownian motion on $[0,1]$]

Coming soon!